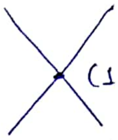
Linear Algebra

- $x + 2y = 3$ ← 1st degree eq^n in 2 dimension

  $2x + 3y = 5$

  $2x + 4y = 6$

  (1 1)

  unique sol^n

  $y = 1$

  $\boxed{y = 1, x = 1}$

- $x + 2y = 3$

  $2x + 4y = 6$

  we can solve these eq^n and so no of sol^{n}s are there to solve. These lines are coincident

  Let $y = k$

  $x = 3 - 2k$

- $x + 2y = 3$ (No sol^n) parallel line

  $x + 2y = 5$

$a_1x + b_1y + c_1z = d_1$ ← 1st degree eq^n in 3 dimension represent plane.

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

we use Rank of a mtx to know whether these 3 eq^n have unique sol^n, multiple sol^n, or no sol^n

Rank of a matrix

Q.1 = $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 2 & 0 \end{bmatrix}$ Find Determinant

3rd row, 2nd column

$$2(-1)^{3+2} \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= -2\{1(2) - 2(0+1) + 1(2-1)\}$$

$$= -2\{2 - 2 + 1\}$$

$$= -2$$

4th row :

$$1(-1)^{4+1} \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{vmatrix} + 2(-1)^{4+3} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= -1\{2(-2-1)\} - 2\{-2(-1-1)\}$$

$$6 - 8 = -2$$

F1 Note (1) $\det(AB) = (\det A)(\det B)$

(2) Adjoint of a matrix

F2 1) $A(\operatorname{adj} A) = (\det A) I$ ← this will be mtx

F3 2) $\det(\operatorname{adj} A) = (\det A)^{n-1}$ ← this will be no.

F4 3) $\operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2} A$ ← this will be mtx

Note: det don't exist for rectangular mtx.

Inverse of a matrix

$AB = BA = I$

To find A^{-1}, $A^{-1} = \frac{\operatorname{adj} A}{\det A}$

$\rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad-bc \neq 0$

Formula

F5 $\rightarrow \det(A^{-1}) = \frac{1}{\det A}$

F6 $A_{m\times n}$ $B_{n\times l}$

F6 $(AB)_{m\times l}$ $\longrightarrow$ $m\times n\times l \rightarrow$ no. of multiplication operations used

F7 $m\times(n-1)\times l \rightarrow$ no. of addition opr used.

$A = \begin{bmatrix} - & - \\ - & - \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} - \\ - \end{bmatrix}_{2\times 1}$

$AB = \begin{bmatrix} - \\ - \end{bmatrix}_{2\times 1}$ 4 multiplication are required

2 add^n are required.

Q2 What is the determinant of this mtx

Q

$$[A]_{m\times n} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$B = n\times m$

Soln

F8 $\boxed{\det(I_m + AB) = \det(I_n + BA)}$ we will use this property

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{4\times1} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}_{1\times4}$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times4} \qquad BA = [4]_{1\times1}$$

$$\det(I_m + AB) = \det P = \det(I_n + BA)$$
$$= \det(1+4)$$
$$= 5$$

$$P = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\underline{\underline{|P| = 5}}$$

Rank of a matrix

② ~~Rank of a mtx~~:

Elementary transformation (we need to know) to get Rank

we have 3 elimⁿ transfⁿ.

1) $R_1 \longleftrightarrow R_2$ interchange of Row $\quad C_1 \longrightarrow C_2$

2) $R_2 \longrightarrow 3R_2$ multiply with a constt $\quad C_2 \longrightarrow 3C_2$

3) $R_2 \longrightarrow R_2 + R_1$ $\quad C_2 \longrightarrow C_2 + C_1$

$R_2 \longrightarrow R_2 + 3$ X (not an elimentary transfⁿ)

$R_2 \longrightarrow R_2 R_1$ X (not)

can E.T can change the deteminant of the mtx (some times Yes
" " No)

can ET can change the Rank of the mtx . (No).

ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$

$|A| = -2$

(1) $R_1 \longleftrightarrow R_2$

$\sim \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2$

deteminant changes the sign

(2) $R_2 \longleftrightarrow 3R_2$

$\sim \begin{bmatrix} 1 & 2 \\ 9 & 12 \end{bmatrix} = -6 = 3 \times -2$

if we multiply one Row with a constt then det also multiplied by that constt

Eg If $\boxed{\det(A_{n\times n}) = P \text{ then } \det(KA) = K^n \det A}$

3) $R_2 \longleftrightarrow R_2 + R_1$

$$\sim \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

$= -2$

4) $R_2 \longleftrightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$= -2$

5) $R_2 \longrightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$= -2$

6) $R_2 \longrightarrow \cancel{5R_2 - 2R_1}$

$\underline{\underline{5}}R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 \\ 13 & 16 \end{bmatrix}$$

$= -10$

$= \underline{\underline{5}}(-2)$

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6.3 Determinant of this mtx

Q

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6 \times 6}$$

— mtx 1

$$\left.\begin{array}{l} R_6 \to R_6 - 2R_1 \\ R_5 \to R_5 - 2R_2 \\ R_4 \to R_4 - 2R_3 \end{array}\right\}$$

due to These E.T det don't change bcz these are not multiplied by k

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

← mtx 2

det = −27

~~Use~~

Eigen values of the mtx after E.T may not same

so for equivalent mtx Eigen values not same

we get mtx 2 is after elimentary transformation.

EV of mtx 2 ≠ EV of metx 1

Here we are reducing mtx 1 into upper triangular mtx so that we can find easily determinant but use those E.T which don't change det

→ For an UT mtx or LTmtx the value of det = product of diagonal elements.

Topic ②

Minor of a matx

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}_{4\times 5}$$

No. of minors of order 4 is 5. (By removing one column i can make a new mtx and det of that mtx called minor.)

No. of minors of order 3 is $^4C_3 \times {^5C_3} = 4\times 10 = 40$

" " " " " 2 is $^4C_2 \times {^5C_2} = 6\times 10 = 60$

No. of minors of " 1 is $^4C_1 \times {^5C_1} = 4\times 5 = 20$

$A_{m\times n}$

1) No. of minors of order r is $^mC_r \times {^nC_r}$.

2) The greatest order minor is min of $\{m, n\}$

Topic ③

Rank of a mtx : exist for square and rectangle mtx.

if Rank is 3 means

1. There exist at least one minor of order 3 which is not 0.

last quesⁿ 3rd order minor total = 40 so at aleast one minor should not be zero.

Rank - order of largest non-zero minor

Q Ex. 4 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{bmatrix}$ Find Rank

$|A| = 0$ so rank not 3

2nd order minor $^3C_2 \times {}^3C_2 = 3\times 3 = 9$ 2x2 minor exist

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4=0$ $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12-12=0$

$\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12-12=0$ $\begin{vmatrix} 4 & 6 \\ 6 & 10 \end{vmatrix} = 40-36 = 4$ so not zero

2x2 minor

So Rank = 2

To find Rank of a mtx also we can use E. Transformation also. any E.T we apply rank is not changed.

Note: - If the mtx $[A]_{4\times4}$ has Rank = 2 it means all the 3rd order and 4th order minor of this mtx are zero.

Echelon Form

mtx said to be Echelon if these two condⁿ satisfied.

1) all "0" rows must present below non zero rows.

2) In the non zero rows before the 1st non zero no., no of zeros must increase

zeros increasing

$$
\begin{array}{l}
\\
\text{1 zero} \\
\text{2 zero} \\
\text{3 zero}
\end{array}
\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}
$$
← In Echelon Form

$$
\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ 0 & b_2 & c_2 & d_2 & e_2 \\ 0 & 0 & 0 & 0 & e_3 \\ 0 & 0 & 0 & 0 & e_4 \end{bmatrix}
$$
← not in echelon form

← bcz no of zeros are not incr.

— same no of zero.

$$
\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ 0 & b_2 & c_2 & d_2 & e_2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$
← Yes In Echelon form.

$$
\begin{bmatrix} 0 & 0 & c_1 & d_1 & e_1 \\ 0 & 0 & 0 & d_2 & e_2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times5}
$$
— Yes in echelon form

Q 5

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$ Find Rank

$R_2 \to 2R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to 2R_4 - 5R_1$

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -3 & -6 & -9 & -12 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - 3R_2$

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← Yes in echelon form

ρ (Rank) = 2 i.e non zero rows

Note*

(1) ET will not affect Rank of mtx

(2) For a null mtx $\rho(O) = 0$

(3) If $A_{m\times n}$ is not a null mtx then min^m ρ can be 1 and max^m ρ can be $\min(n,m)$

If $A_{m\times n} \neq 0$

$$\boxed{\rho_{min} = 1 \quad , \quad \rho_{max} = \min(n,m)}$$

(4) The rank of a non singular mtx of order n is "n".

(5) " " " " Singular mtx of " "n" is less than "n".

Impr. (6) If all minor of order "r" of A are zero then $P(A)$ is less than r

(7) If all the elements of are fixed constt and same and $K \neq 0$ then $P(A)=1$

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

$P(A)=1$

All elements th are fixed constt

(8) $P(I_4) = 4$ $\quad P(I_n) = n$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(9) The rank of

$$A = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix}_{6\times 6}$$

all are "0".

all are "0".

$P(A) = 4$

10). If the rank of a row mtx is 0 or 1.

$A = [0\ 0\ 0\ 0]$

$P(A) = 0$

$B = [1\ 1\ 0\ 1]$

$P(B) = 1$

Imp. 11. Rank of a column mtx is 0 or 1.

Row Echelon form - $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}_{3\times 1} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ — Rank(1) of this

we know max^m Rank can be

$P(max) = min(m \times n)$

$= min(3\times 1)$

$P(max) = 1$

the rank can't be greater than 1 bcz (3×1), 1 is small.

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ rank is (0)

(11) If $A_{m\times 1}$ & $B_{1\times n}$ are two non zero mtx then $\rho(AB)$ is $\underline{1 \text{ (always)}}$

a) m

b) n

c) $\min\{m,n\}$

d) 1 ✓

o $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$ $B = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}_{1\times 4}$

$$(AB)_{3\times 4} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 6 & 8 & 10 \\ 6 & 9 & 12 & 15 \end{bmatrix}_{3\times 4}$$

$\rho(AB) = 1$

o $A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 6 & 8 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$ $\rho(AB) = 1$

rank of AB can never be zero bcz mtx are nonzero.

(12) If $A_{m\times 1}$ is a non zero mtx then $\rho(AA^T) = 1$, $\rho(A^TA) = 1$

$A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$

$\rho(AA^T) = 1$

13. $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

If $\rho(A_{7\times 4}) = 5$ not possible

$\rho(A_{7\times 4}) = 3$ $\rho(B_{4\times 5}) = 2$

$\cancel{\rho(AB)_{4\times 5}}$

than $\rho((AB)_{7\times 5}) \leq 2$. even though $(AB)_{7\times 5}$ is of 7×5 but still rank ≤ 2.

14. ~~Rai~~ $\rho(A) = \rho(AA^T)$

15.) $\rho(A+B) \leq \rho(A) + \rho(B)$

16. $\rho(A-B) \geq \rho(A) - \rho(B)$

17. $\rho(A) = \rho(A^T)$

18. If $\rho(A_{n\times n}) = n$ then $\rho(\text{adj}A) = n$

Since $\det(\text{adj}A) = (\det A)^{n-1}$

~~19~~.

19. If $\rho(A_{n\times n}) = n-1$ then $\rho(\text{adj}A) = 1$

ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{pmatrix}$$

$\rho(A) = 2$

$\text{Adj } A = \begin{bmatrix} \\ \ldots\ldots \end{bmatrix}$ ← last row will be 0

20. If $\rho(A_{n\times n}) = n-2$ then $\rho(\text{adj}A) = 0$

if $n = 4$ rank $= 2$

$\begin{bmatrix} \\ 0\ 0\ 0\ 0 \end{bmatrix}$ ← so all 3rd order and 4th order miners are 0. and when we calc adj A, it will contain 4th order miner so $\rho(\text{adj}A) = 0$.

21) Nullity of A = No. of column of A − $\rho(A)$

$A_{4\times4}$ and $\rho(A) = 3$ so nullity = $4 - 3 = 1$

$A_{4\times4}$ and $\rho(A) = 4$ nullity = 0

If det $A \neq 0$ then nullity of $A = 0$

If det $A = 0$ then nullity of $A \neq 0$

System of Equations

$a_1x + b_1y + c_1z = d_1$ ← its an eq^n in 3 dimensional plane ← 1st order non homogen eq^n

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$AX = B$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Augumented mtx

$$A|B = \left[\begin{array}{ccc|c} A_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \end{bmatrix}$$

i) if $\rho(A|B) = \rho(A) = n$ (no. of variables) ⇒ Unique solution

ii) If $\rho(A|B) \neq \rho(A)$ No sol^n or inconsistent system

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & 0 & d_3 \end{bmatrix}$$

$0 = d_3$ ← this is not possible thats why no sol^n.

number

iii) If $\rho(A|B) = \rho(A) = r < n$ (no. of variables)

⇒ ∞ solutions. in $(n-r)$ independent ~~sol^n~~ variable.

let $z = K$
$y = f(K)$
$x = f(K)$

Choos $K = \infty$ real no. so ∞ solution.

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho = 1$

$n - r = 3 - 1 = 2 =$ no. of independent variable.

$z = K$
$y = l$
$x = (K, l)$

It is not necessary that we take $z = K$ and $y = l$
x can also be independent

Q6 For what values of a and b the system of eq^{n}s have
(1) No sol^n
(2) unique sol^n
(3) ∞ sol^n.

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{bmatrix}$$

$R_3 \to R_3 - (R_1 + R_2)$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 0 & 0 & a-8 & b-15 \end{bmatrix}$$

By sir

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{bmatrix}$$

1) if $a = 8$ and $b \neq 15$ No sol^n

2) if $a \neq 8$ "b" can be any no. unique sol^n

3) if $a = 8$ and $b = 15$ ∞ solutions.

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let $z = k$ ~~a = 8 b = 15~~

$$y + 2z = 3$$
$$y = 3 - 2z$$
$$y = 3 - 2k$$

$$x + 2(3-2k) + 3k = 6$$
$$x = k$$

Q7 By solving the system of Eqns

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

the value of x = ___

Soln bcz we are asked for the numerical value of x so unique soln

$$A_{4\times3} \, X_{3\times1} = B_{4\times1}$$

$$[A|B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 3R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$R_4 \longrightarrow R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_3 \rightarrow 7R_3 - 6R_2$

$R_4 \rightarrow 7R_4 - 3R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no need to solve there will be 0. bcz then only the unique soln will be there

$5z = 20 \qquad z = 4$

$-7y + 5z = -8$

$y = 4$

$x + 2(4) \cancel{\;} - 4 = 3$

$x = -1$

case II Let for some other system of eqn

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

← This in Echelon form

Ask?

$\rho(A) = 3 \qquad \rho(A|B) = 4$

$\rho(A) \neq \rho(A|B) \Rightarrow$ no solution (Is it correct)

If we have just two eqn

Case III $x+2y-z=3$ these are eqn of planes.

$3x-y+2z=1$

Let this is our Augumented mtx

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

two planes can't intersect at a unique point so no unique soln

$A_{m\times n} X_{n\times 1} = B_{m\times 1}$

m = no. of equations n = no. of variables

1) If $m \geq n$ all three cases ~~possible~~ can be possible

2) If $m < n$ no unique soln, It may have ∞ solution or no solutions

Homogenious System

homogenious eqn. $\rightarrow$ $a_1x^{(1)} + b_1y^{(1)} + c_1z^{(1)} = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

all variable have same power so called homogenous.

In $a_1x^{(1)} + b_1y^{(1)} + c_1z^{(1)} = d_1w^{(0)}$ $\leftarrow$ nonhomogenious eqn

x, y, z power is 1 but power of w is 0. so non-homogenous eqn.

$$AX = 0$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A_1 = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}$$

(1) case I — If $\rho(A) = n$ (no. of variable) unique soln ~~and~~ or zero soln or trivial soln only.

अगर unique soln होगा तो वो trivial soln ही होगा

~~$\rho(A)$ & r ...~~

case II — $\rho(A) = r < n$ then

infinite soln in $(n-r)$ independent ~~soln~~ variable

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} z &= k \\ y &= f(k) \\ x &= (k) \end{aligned}$$

$$A_1 = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 1 \qquad n - r = ~~2~~ ~~1~~ 2$$

$3 - 1 =$

Q 8 The System of eqn

$$x+y+z=0$$
$$(a+1)y+(a+1)z=0$$
$$(a^2-1)z=0$$

have non trivial soln in two independent variable. then $a=$ ____

Soln

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & a+1 & a+1 \\ 0 & 0 & a^2-1 \end{bmatrix}$$

when $a=1$ system has non trivial soln but in one independent variable

$$n-r=2$$
$$3-\rho(A)=2$$
$$\rho(A)=1$$

soln if $a=\underline{\underline{-1}}$ Ans.

Q 9 The system of Eqns

$$AX=0$$

When A is an nxn mtx an have non-trivial soln if

a) $\rho(A)=n$

$n-r \neq 0$

b) $\det A \neq 0$

c) A is non singular

d) Nullity of $A \neq 0$

soln d ✓ $\Rightarrow$ $\det A=0 \Rightarrow \rho(A)<n$

nullity = no. of column of A − rank of A

Eigen Vectors

$$|A - \lambda I| = 0$$

$P(\lambda) = 0$ = Characteristic ~~no~~ equation

roots of $P(\lambda)$ = called Eigen values

Ex = $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda) - 4 = 0$$

$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow$ Char' eqn

$$\lambda^2 + 6\lambda + \lambda + 6$$

$$\lambda(\lambda+6) + 1(\lambda+6)$$

$$\lambda = -1, \lambda = -6$$

Eigenvalues

2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

$$(1-\lambda)(2-\lambda) - 6$$

$$2 - 3\lambda + \lambda^2 - 6$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\frac{3 \pm \sqrt{9 + 16}}{2}$$

$$\lambda = \frac{3 \pm 5}{2},$$

$$\lambda = -1, 4$$

Properties :-

(1) Sum of E·Values of A = Trace of A = $(-1)^{n-1} \times$ coeff of λ^{n-1} Imp^r.

if coff λ^n is 1. ***

$$\lambda^n + K_1 \lambda^{n-1} + \cdots + K_n = 0$$

$$\text{Trace} = (-1)^{n-1} \times K_1$$

(2) product of Evalues of A = det A = $(-1)^n \times$ constt term ~~of~~

if coff. of λ^n is 1., in char^c eq^n.

if det of a mtx is 0 then atleast one E·Value is 0.

3) The E·value of a symm^r mtx are purely real.

(4) The E value of a skew symm^r mtx ~~or~~ are either zero or purely imaginary

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \leftarrow \text{Skew symm}^r \text{ (even ordered)}$$

$$\begin{pmatrix} 0-\lambda & 2 \\ -2 & 0-\lambda \end{pmatrix} \Rightarrow \quad \lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

The determinant of an odd ordered skew symm^r mtx is zero since atleast one of the eigen value is 0.

(5) The E·values of an upper triangular mtx or a lower triangular mtx are its diagonal element only but its converse need not be true.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 4 & 5-\lambda \end{bmatrix}$$

converse need not be true

$$(1-\lambda)(3-\lambda)(5-\lambda) = 0$$

1, 3, 5

6. If A has E. values $\lambda_1\ \lambda_2 \ldots \lambda_n$ then

(i) A^2 has E. values $\lambda_1^2\ \lambda_2^2 \ldots \lambda_n^2$

(ii) A^r has E. values $\lambda_1^r, \lambda_2^r \ldots \lambda_n^r$

(iii) A^{-1} has " $\frac{1}{\lambda_1}\ \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_n}$

(iv) KA has " " $K\lambda_1, K\lambda_2, \ldots, K\lambda_n$

(v) $A-KI$ " " $\lambda_1 - K, \lambda_2 - K, \ldots \lambda_n - K$.

(vi) adj A " " $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2} \ldots \frac{|A|}{\lambda_n}$

dont read it mod A. read it det A.

$$A\,(\text{adj} A) = (\det A) I$$

$$\text{adj } A = (\det A)\, A^{-1}$$

Q10 If A has eigen 2, 3, 1 then $A^2 + 3A + 2I$ has eigen value.

Solⁿ ~~$2^2 + 3\times 2 + 2$~~

~~$4 + 6 + 2 = 12$~~

~~$1^2 + 3\times 1 + 2 =$~~

$A \to$	2	3	1
$A^2 \to$	4	9	1
$3A \to$	6	9	3
$2I \to$	2	2	2

$A^2 + 3A + 2I \to$ 12, 20, 6

Q11 $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Find Evalues.

A matrix $A_{n\times n}$ having all elements 1 has Characteristic equation

$$\boxed{P(\lambda) = \lambda^n - n\lambda^{n-1}}$$

$(1-\lambda)^2 - 1$

$1+\lambda^2-2\lambda-1=0$

$\lambda^2 = 2\lambda$

$\lambda(\lambda-2)=0$

$\lambda = 0, 2$

Formula.

$$\boxed{P(\lambda) = \lambda^n - n\lambda^{n-1} = 0}$$

$= \lambda^2 - 2\lambda$

$= \lambda^2 - 2\lambda = 0$

Q 12 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$ Find Evalues.

$P(\lambda) = \lambda^n - n\lambda^{n-1} = 0$

$= \lambda^3 - 3\lambda^2$

$\cancel{(\lambda-1)}^3$

$\lambda^3 - 3\lambda^2 = 0 \quad \lambda = 0, 0, 3$

Q 13 $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ 1 & & & \\ 1 & 1 & 1 \cdots & 1 \end{bmatrix}_{n\times n}$

Evalues $=$ 0 0 0 .. n-1 times, n

Q14 The product of non-zero eigenvalues of

$$A = \begin{array}{c|c} & 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{bmatrix} \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Soln $(1-\lambda)\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \\ 1 & 0 & 0 & 0 \end{vmatrix}$

Ans.

$$(1-\lambda)^2 \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

(003) (0003)

(003) $\{(1-\lambda)^2 - 1\} = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda = 0, 2$

0 0 0 2 3

~~0 0 0, 0 0 6~~

$((1-\lambda)^2 - 1)(\lambda^3 - 3\lambda^2) = 0$

$\boxed{\lambda = 0\ 0\ 0\ 2\ 3}$

Sol^n

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$= (1-\lambda)^2(\lambda^3 - 3\lambda^2) - \lambda^3 - 3\lambda^2$

$= (\lambda^3 - 3\lambda^2)((1-\lambda)^2 - 1)$

$= \lambda^2(\lambda - 3)(1 + \lambda^2 - 2\lambda - 1)$

$= \lambda^2(\lambda-3)(\lambda^2 - 2\lambda)$

$= \lambda^2(\lambda-3)\lambda(\lambda-2)$

$\lambda = 0, 0, 0, 2, 3$

$2 \times 3 = \underline{\underline{6}}$

$$1-\lambda \begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} + (-1)^{1+5} \times 1 \begin{vmatrix} 0 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$(1-\lambda)^2 \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} + (-1)^{4+1} \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

Eigen vector

The non-trivial sol^n of the homogenous system $(A - \lambda_r I)X = 0$ or $AX = \lambda_r X$ are called Eigenvectors corresponding to the eigen value λ_r of the mtx A

Q15 Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ Find Eigen vector

$\lambda = 4, -1$

E. vector corresponding to 4

$$(A - 4I)X = 0$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \longleftrightarrow R_2 + R_1$

$$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x + 2y = 0$$

Let $y = K$ $\quad x = \frac{2K}{3}$

$$\begin{bmatrix} \frac{2K}{3} \\ K \end{bmatrix}$$

how can generate ∞ no of E. vector for ∞ real values.

E. vector corresponding to (-1)

$$(A + I)X = 0$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x + 2y = 0$

let $y = 1$

$x = -1$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Q16 $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ which of the following is not an eigen vector.

a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$

c) $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

d) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Solⁿ (d) not an E-vector

Q[17] Which of the following is an E·vector of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

a) $(1\ 1\ 1)^T$

b) $(1\ 0\ 1)^T$

c) $(0\ 1\ 1)^T$

d) $(1\ 0\ 0)^T$

Soln.

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$(2-\lambda)\{(2-\lambda)^2\}$$

$$\lambda = 2, 2, 2$$

$[A-\lambda I]$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$z = k$~~

this is echleon $P = 2$ so $n-P = 3-2 = 1$ independent variable

either x or y or k any can be independent it depends on situation.

From this eqn $z = 0$

From this eqn $y = 0$

let $x = k$

$$\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$$

Q. 18 Which of the following is E.vector

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) $(1\ 1\ 1)^T$
b) $(1\ 0\ 1)^T$
c) $(0\ 1\ 1)^T$
d) $(1\ -1\ 1)^T$

Soln

$\lambda = 2\ 2\ 2$

$(A - 2I)X = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho = 1 \qquad n = 3$

$3 - \rho = 2$

$3 - 1 = 2 \Leftarrow \Rightarrow$ 2 independent variable

$0x + y + z = 0$

$y = -z$

let ~~y = P~~

$\boxed{z = k}$ ⇐ 1st variable choosen independently

$\boxed{y = -k}$

$\boxed{x = l}$ ⇐ 2nd variable choosen independently

$$\begin{bmatrix} l \\ -k \\ k \end{bmatrix} \simeq \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$ so option d

Q 19 Which of the following is not an E-vector of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) $(1\ 1\ 1)^T$
b) $(1\ 2\ 2017)^T$
c) $(0\ 0\ 1)^T$
d) $(0\ 0\ 0)^T$

Soln $\lambda = 2, 2, 2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho = 0 \quad n = 3$

$n - \rho = 3 - 0 = 3$ can be choosen in depen

EV is a non trivial solⁿ of Homogenous Soln

$\{A - \lambda I\}$ ~~is the homogeneous~~

let $x = k$, $y = l$, $z = m$

$$\begin{bmatrix} k \\ l \\ m \end{bmatrix} \leftarrow \text{eigenvector.}$$

$[A - \lambda I]X = 0$ is homogeneous Soln

thats why $[0\ 0\ 0]^T$ can never be EV bcz eigenvector

Q 20 The E-vector $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix}$ then $a + b =$

$\lambda = 1, \quad \lambda = 2$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$y = 0$
$x = 0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By Sir $\quad AX = \lambda X$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 1\begin{bmatrix} 1 \\ a \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 2\begin{bmatrix} 1 \\ b \end{bmatrix}$$

$1 + 2a = 1$

$\cancel{2a = a}$

$a = 0$

$1 + 2b = 2$

$0 + 2b = 2b$

$b = 1/2$

$$a + b = \frac{1}{2}$$

Note*

(1) The E. vectors corresponding to distant E. values of a mtx are linearly independent

(2) If λ_r is an E. value of a mtx A then for the homogeneous system $[A - \lambda_r I]X = 0$ always non trivial sol^n exist ie $X \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(3) E. vectors of $A, A^2, KA, \text{adj } A, A^{-1}, A - KI$ are all same
(Eigen values are ~~same~~ not same.)

(4) E vector of A and A^T are not the same
but Eigen values of A and A^T are same

(5) The E. Vector corresponding to distant eigen values of a symmetric matrix are always orthogonal to each other

$A_{3\times 3}$ = Symmetric mtx

$\lambda_1 \neq \lambda_2$ $\quad$ so X_1 and X_2 are linearly independent

$$\begin{array}{cc} X_1 & X_2 \\ \begin{bmatrix} \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \\ \end{bmatrix} \end{array} \qquad X_1^T X_2 = 0$$

Orthogonal (Tricks sembly).

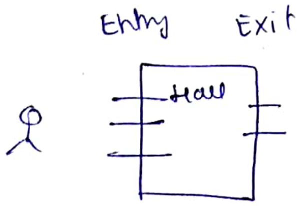
Probability

Arrangement & Selection
$\Downarrow$ $\Downarrow$
Permutation combination

Fundamental Principle of Counting (FPC)

1. Fundamental Principle of Counting (FPC)

Q 3 entry gates and 2 exit gates so how many ways a person can travel



3×2

2 nds, fill 10 — 9

$\underline{\ \ }$ $\underline{\ \ }$
9 8

0, 1, 2, 3 given mtx

$\begin{bmatrix} - & - \\ - & - \end{bmatrix}_{2\times 2}$ 4^4

$\begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}_{4\times 1}$

$\begin{bmatrix} - & - & - & - \end{bmatrix}_{1\times 4}$

$= 3 \times 4^4 \leftarrow$ ways.

3 B 3 G 3 G sit together

$$\frac{\lfloor 4 \lfloor 3}{\lfloor 6}$$

2. ^nC_r ^nP_r

$$^nC_r = \frac{n!}{r!\,(n-r)!} \qquad ^nP_r = \frac{\lfloor n}{\lfloor n-r}$$

a, b, c select two letter

Combination

ab

bc

ac

$^3C_2 = 3$

Permutation

a b

b a

b c

c b

c a

a c

$^3P_2 = 6$

(1) 10 friend want to shake hand $^{10}C_2$

(2) " " want to send email $^{10}P_2$

(3) 10 station, person want to travel so no. of tickets $^{10}P_2$

4) 10 teams, How many inning $^{10}P_2$

How may match $^{10}C_2$

3. $^nP_n = \lfloor n$

^nP_n involves Fundamental Principle.

4. $\underbrace{^nC_r}_{\text{r things are selected}} \times \underbrace{\lfloor r}_{\text{arranged in } r!} = \underline{^nP_r}$

5) $\dfrac{\lfloor n}{\lfloor k}$ K-same things , arrange n things where k same things

a, b, c, d, e, e

$\dfrac{\lfloor 6}{\lfloor 2}$

6) $\dfrac{\lfloor m+n+l}{\lfloor m \lfloor n}$ m things are sam n-things are same

l things are all different

a a b c d e e e

$\dfrac{\lfloor 8}{\lfloor 2 \lfloor 3}$

7) Circular arrangement

$\lfloor n-1$

8) $^nC_r = {}^nC_{n-r}$

50 chocolate ~~select~~ select 45 Chocolate

$^{50}C_{45}$ or for easyness $^{50}C_5$

Do

Ex 1 a a b c d e e e — How many ways we can select 3 letter, and How many ways we select Arrange 3 letters.

3 same → $1 \times 1 = 1$

2 same 1 diff → $\frac{8 \times \lfloor 3}{\lfloor 2} = 24$ → a a < b, c, d, e

3 different → $10 \lfloor 3 = 60$

(60/85) Ans.

e e < a, b, c, d

$1 \times {}^4C_1 + 1 \times {}^4C_1 = 8$

$^5C_3 = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times}{2} = 10$

	Selection	Arrangement
3 same	1	1
2 same 1 diff	8	24
3 differ	10	60

Q2 3 pens are to be brought to an interview by a candidate out of available colours Red, Blue, Black, green. What is the probability that a person to bring all the 3 pens are of different colours.

Solⁿ. 3 same or 2 same 1 diff or all diff

Here is selection process only.

3 same → 4 cases.

2 same 1 diff → 2 ways to do this

$^4C_1 \times {}^3C_2 = 12$ → 3 colours में से कोई 2 colour choose करें

or

$^4C_2 \times 2 = 12$

12 case

all diff → $^4C_3$

4 cases

all cases = 20

So $\frac{\text{all different case}}{\text{all case}} = \frac{4}{20} = \frac{1}{5} = 0.2$

Arrangement of persons

Do

03 6B and 6G are to be arranged in a row at random what is the probability that

(1) All the girls to sit together

(2) No two girls to sit together

(3) all the girls are not to sit together

(4) No two boys and no two girls will sit together.

Soln

g g g g g g

~~||||||~~ | | | | | |

1 Girl & 6 Boys

(1) $= \dfrac{7!\,6}{12!} = \dfrac{\lfloor 7 \times \lfloor 6}{\lfloor 12}$ — group में girls खुद को interchange करा सकती हैं 6! के type से

(2) 6 Boys

_ X _ X _ X _ X _ X _ X _

1 2 3 4 5 6 7

7 gap

out of available 7 gaps select any 6

interchange girls

$$\frac{\lfloor 6 \times {}^7C_6 \times \lfloor 6}{\lfloor 12}$$

(3) $1 - \dfrac{\lfloor 7\ \lfloor 6}{\lfloor 12}$

(4) first arrang Boys in 6!

let exclude 7th gap.

_ X _ X _ X _ X _ X _ X (_

if we remove the gap at middle then two Boys will be same so remove 7th gap and 1st gap.

_) X _ X _ X _ X _ X _ X _

$$\frac{\lfloor 6\ \lfloor 6 + \lfloor 6\ \lfloor 6}{\lfloor 12}$$

Same que as before

Q 6 Boys and 5 girls are to be arranged in a row at random

(I) ~~$G_1 G_2 G_3 G_4 G_5$~~ $B_1 \ B_2 \ B_3 \ B_4 \ B_5 \ B_6$

G

$$\frac{\lfloor 7 \lfloor 5}{\lfloor 11}$$

(II) _ X _ X _ X _ X _ X _ X _

$$\frac{\lfloor 6 \times {}^7C_5 \times 5!}{\lfloor 11}$$

Arrange boys first so 7 gap and 5 girls

interchang girls

(III) $1 - \frac{\lfloor 7 \lfloor 5}{\lfloor 11}$

(iv) _ X _ X _ X _ X _ X _ X _

1 2 3 4 5

Gap1 G2 G3 Gap4 Gap5 Gap6 Gap7

First arrange Boys

we need to remove Gap1 & Gap7

$$= \frac{\lfloor 6 \lfloor 5}{\lfloor 11 \ \lfloor 12}$$

Q 6 boys and 4 girls are arranged in a row at random

(1) $\cancel{\frac{7!\,4!}{\lfloor 10}}$ $\frac{7!\,4!}{10!}$

(2) $\frac{\lfloor 6 \; {}^{7}C_{4} \times 4!}{\lfloor 10}$

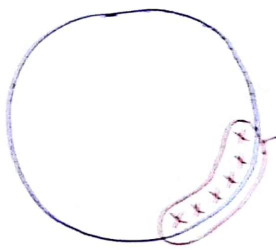
3) $\frac{\lfloor 10 - \lfloor 7 \lfloor 4}{\lfloor 10}$

4) $\frac{0}{\lfloor 10}$

Q 6 Boys and 6 girls are to be arranged in a circular order at random

Sol^n 12 person can be arranged in $(12-1)! = \lfloor 11$

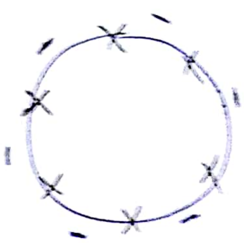
1) $\frac{\lfloor 6 \lfloor 6}{\lfloor 11}$



ii) $\frac{\lfloor 5 \times \lfloor 6}{\boxed{\lfloor 11}}$

6 boys can be arr in 5! way

6 girls in 6 gaps can be in 6! ways



6 gaps

(iii) $\frac{\lfloor 11 - \lfloor 6 \lfloor 6}{\lfloor 11}$

(iv) same as (ii) bcz no extra gaps are available here.

$\frac{\lfloor 5 \lfloor 6}{\lfloor 11}$

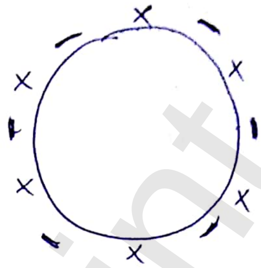
Q 6 Boys and 5 girls in circular

(i) ~~5 girls~~ 6 Boy

1 + 6 = 7

$$\frac{\lfloor 6 \lfloor 5}{\lfloor 10}$$ 5 girls can be interchanged

(ii) $$\frac{\lfloor 5 \ {}^{6}C_{5} \times \lfloor 5}{\lfloor 10}$$



(iii) $$\frac{\lfloor 10 - \lfloor 6 \lfloor 5}{\lfloor 10}$$

iv) $$\frac{0}{\lfloor 10}$$

Q 6 same apples and 6 girls are to be arranged in row in at random.

(i) $$\frac{\left(\frac{\lfloor 7}{\lfloor 6} \times \lfloor 6\right)}{\frac{\lfloor 12}{\lfloor 6}}$$ girls can be interchanged

$\frac{\lfloor 12}{\lfloor 6}$ ← bcz 6 apples are same.

(ii) $$\frac{1 \times {}^{7}C_{6} \ \lfloor 6}{\frac{\lfloor 12}{\lfloor 6}}$$

_ X _ X _ X _ X _ X _ X _

(iii) $$\left(\frac{\frac{\lfloor 12}{\lfloor 6} - \lfloor 7}{\frac{\lfloor 12}{\lfloor 6}}\right)$$

Tossing of coins

Coin	Total cases
1	2
2	2^2
3	2^3
....	....
n	2^n

Do Q A coin is tossed until it shows same faces in consecutive ~~thoroughs~~ throws what is the prob. of successly tossing the coin not more than 4 times.

Sol^n

2 times $\longrightarrow$ $\dfrac{2}{2^2}$ HH or TT

2^2 ← Total case

+

3 times $\longrightarrow$ $\dfrac{2}{2^3}$ HTT or THH

+

4 times $\longrightarrow$ $\dfrac{2}{2^4}$ HTHH or THTT

$$\Rightarrow \frac{2}{2^2} + \frac{2}{2^3} + \frac{2}{2^4}$$

$$\Rightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Q if not more than 3 times

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

← when 5 times

$$\frac{3}{4} < \frac{7}{8} < \frac{15}{16}$$

as no. of times Tes. prob Tes.

Q step I - Toss a coin two times

step II - If it is HT then output is Y

step III - If it is HH or TT then O/p is N

step IV - If it is TH then go to step I.

then what is the probability for output to be Y.

Soln

$a + ar + ar^2 \ldots$

$S = \frac{a}{1-r} \qquad |r| < 1$

HT + TH HT + TH TH HT

1st 1st 2nd 1st 2nd 3rd

$= \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} + \ldots$

$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3$

$a = 1/4 \qquad r = 1/4$

$= \frac{1/4}{1 - 1/4} = \frac{1}{4} \frac{4}{3} = \frac{1}{3}$

DO

Q prob to get ~~reser~~^{r} reservation confermed = 0.3

avg no. of attempts required to confermed a ticket.

Soln mean $= \sum x_i p_i$

x = no. of attempts

x	1	2	3
$P(x)$	0.3	$(0.7)(0.3)$	$(0.7)^2(0.3)$

$\sum_i x_i p_i = 1(0.3) + 2(0.7)(0.3) + 3(0.7)^2(0.3) + \cdots\cdots$

$= 0.3\left[1 + 2(0.7) + 3(0.7)^2 + \cdots\cdots\right]$

$*$ A $*$

$$\boxed{(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3} \leftarrow \text{Remb}^r \text{ this}$$

$= 0.3\left[1 - 0.7\right]^{\cancel{-2}\, -2}$

$= 0.3[0.3]^{-2}$

$= \dfrac{0.3}{(0.3)^2} = \dfrac{1}{0.3} = \cancel{3.33}\ 3.33$ //

Q A & B simultaneously toss a coin until one of them get head if A starts the game what is prob^r for B to win

Soln $P(A\text{win}) = \frac{1}{2}$ $P(B\text{win}) = \frac{1}{2}$

$= \bar{A}B + \bar{A}\bar{B}\bar{A}B + \bar{A}\bar{B}\bar{A}\bar{B}\bar{A}B$

$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} +$

$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3$ $S_n = \dfrac{a}{1-r}$ $|r| < 1$

$a = 1/4$ $r = 1/4$

$= \dfrac{1/4}{1 - 1/4} = \dfrac{1.4}{4.3} = \dfrac{1}{3}$

For A to win Prob $= 1 - \frac{1}{3} = \frac{2}{3}$

Person who starts the game has more chances

Question : A coin is tossed until Head appears. What is prob of success in even no. of tosses.

Soln

Answer = $\frac{1}{3}$

Rolling of dice

1 dice $\longrightarrow$ 6

2 dice $\longrightarrow$ 6^2

3 dice $\longrightarrow$ 6^3

n dice $\longrightarrow$ 6^n

Easy.

Q A & B simultaneously roll a die until one of them gets 5. if A starts the game what is the prob for B to win

$P(A) = \frac{1}{6}$ $\quad P(\bar{A}) = \frac{5}{6}$

$\bar{A}B + \bar{A}\bar{B}\bar{A}B + \bar{A}\bar{B}\bar{A}\bar{B}\bar{A}B$

$\frac{5}{6}\times\frac{1}{6} + \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} + \quad \ldots$

$$= \frac{\frac{5}{36}}{1-\frac{25}{36}} \qquad \frac{a}{1-r}$$

$P(B) = \frac{5}{11}$

Q 2 dices is rolled until the sum is 5 or 7 what is the prob^r for sum as 5 before sum as 7.

Sol^n 5 — (1 4) (4 1) (2 3) (3 2)

$$P(5) = \frac{4}{36} = \frac{1}{9}$$

P(7 ⟶ (1 6) (6 1) (2 5) (5 2) (3 4) (4 3)

$$P(7) = \frac{6}{36} = \frac{1}{6}$$

~~P(5)~~ or

$$\boxed{P(S \neq 5 \cap S \neq 7) = 1 - \frac{10}{36} = \frac{26}{36}}$$

S=5 (1st) + S≠5, S≠7 (1st) S=5 (2nd) + S≠5, S≠7 (1st) S≠5, S≠7 (2nd) S=5 (3rd) + . . .

$$= \frac{4}{36} + \frac{26}{36} \times \frac{4}{36} + \frac{26}{36} \times \frac{26}{36} \times \frac{4}{36} + \cdots$$

$$= \frac{4/36}{1 - \frac{26}{36}} = \frac{4}{10} = 0.4$$

Q There are two biasiased dices of which 1st dice should an even no twice as frequently as an odd number 2nd dice shows the number 5 thrice as frequently as any other number If these two dice are rolled together what is the prob for sum as 10

	dice 1		dice 2	
1	k	— 1/9	l	— 1/8
2	2k	— 2/9	l	— 1/8
3	k	— 1/9	l	— 1/8
4	2k	— 2/9	l	— 1/8
5	k	— 1/9	3l	— 3/8
6	2k	— 2/9	l	— 1/8

$9k = 1$ $\qquad$ $8l = 1$

$k = 1/9$ $\qquad$ $l = 1/8$

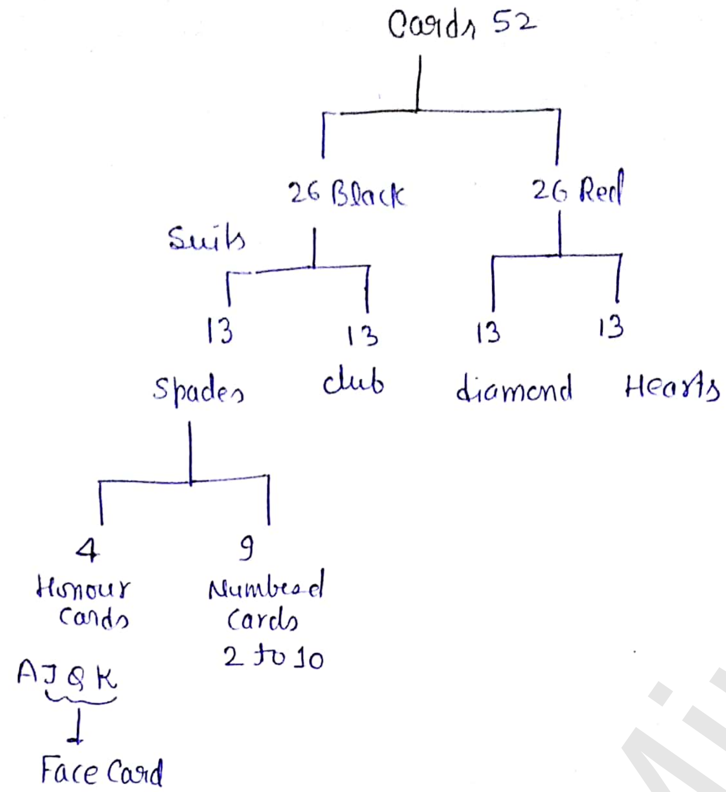
$$\text{Sum}(10) = (4\,6) \text{ or } (6\,4) \text{ or } (5\,5)$$

$$= \left(\frac{2}{9} \times \frac{1}{8}\right) + \left(\frac{2}{9} \times \frac{1}{8}\right) + \left(\frac{1}{9} \times \frac{3}{8}\right)$$

$$= \frac{2}{72} + \frac{2}{72} + \frac{3}{72}$$

$$= \frac{7}{72}$$

Q ~~Taking a card from a deck pack~~ :



Q Two cards are drawn at random from a pack, what is the prob that both must belong to same suit

b) both must belong to diff suite

(a) $\dfrac{{}^{13}C_2 \times {}^{4}C_1}{{}^{52}C_2}$

Choosing 2 cards out of 13

ways of choosing one suit out of 4.

b) dont do 1 − P(as in case a)

$\dfrac{{}^{4}C_2 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_2}$

choosing two suits

Choosing one card from one suit

When 3 cards are drawn

(i) all card belong to same suite

$\dfrac{{}^{4}C_1 \times {}^{13}C_3}{{}^{52}C_3}$

ways of choosing one suite

3 card from same suite

b) belong to diffnt suite

$\dfrac{{}^{4}C_3 \; ({}^{13}C_1)^3}{{}^{52}C_3}$

is the ways of choosing 3 suits out of 4

(c) When 4 cards are drawn

i) $\frac{{}^4C_1 \times {}^{13}C_4}{{}^{52}C_4}$ ii) $\frac{{}^4C_4 \times ({}^{13}C_1)^4}{{}^{52}C_4}$

d) When 5 Cards drawn

(i) $\frac{{}^4C_1 \times {}^{13}C_5}{{}^{52}C_5}$ (ii) $\frac{0 \times ({}^{13}C_1)^5}{{}^{52}C_5}$

0 bcz we cant choose 5 cards differently from 4 suits atleast 2 cards will be from same suite.

${}^4C_5 \Rightarrow$ not possible.

Q A card is drawn at random from a pack of cards what is the probability that it is

(1) A king or a red card
(2) A spade or a face card
(3) A diamond or Honour card
(4) A king or queen

Soln (1) $\frac{{}^4C_1 + {}^{26}C_1}{{}^{52}C_1}$

(2) $\frac{{}^{13}C_1 + {}^{12}C_1}{{}^{52}C_1}$

Addition Theorem :-

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If i do like this what mistake i am doing.

By Sir

(1) $P(K \cup R) = P(K) + P(R) - P(K \cap R)$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

ii) $P(S \cup F) = P(S) + P(F) - P(S \cap F)$

$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$

(iii) $P(D \cup H) = P(D) + P(H) - P(D \cap H)$

$= \frac{13}{52} + \frac{16}{52} - \frac{4}{52}$

(iv) $P(K \cup Q) = P(K) + P(Q) - P(K \cap Q)$

$= \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$

$=$

Multiplication Theorem or
Conditional probability:

$$P(A \cap B) = P(A)\, P\left(\frac{B}{A}\right)$$

or

$$P(B)\, P(A/B)$$

Q Two Cards are drawn one after the other without replacement What is the probability that

(i) 1st card is king & 2nd card is queen

(ii) 1st card is king & 2nd card also a king

(iii) 2nd card is a king

Soln i) $P(K_1 \cap Q_2) = P(K_1) \times P\left(Q_2/K_1\right)$

$= \frac{4}{52} \times \frac{4}{51}$

ii) $P(K_1 \cap K_2) = P(K_1) \times P\left(\frac{K_2}{K_1}\right)$

$= \frac{4}{52} \times \frac{3}{51}$

2. $P(K_1) \times P\left(\frac{K_2}{K_1}\right) + P(NK_1) \times P\left(K_2/NK_1\right)$

Prob of 2nd card to be king knowing that 1st card is non king.

$= \frac{4}{52} \times \frac{3}{51} + \frac{48}{52} \times \frac{4}{51}$

Independent Events :- Two events A and B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$

i.e happening of one event has no effect on other

Q In the above problem if with replacement

(i) $P(K_1 \cap Q_2) = P(K_1) \times P(Q_2/K_1) = \frac{4}{52} \times \frac{4}{52}$

(ii) $P(K_1 \cap K_2) = P(K_1) \times P\left(\frac{K_2}{K_1}\right) = \frac{4}{52} \times \frac{4}{52}$

(iii) P $1 \times \frac{4}{52}$

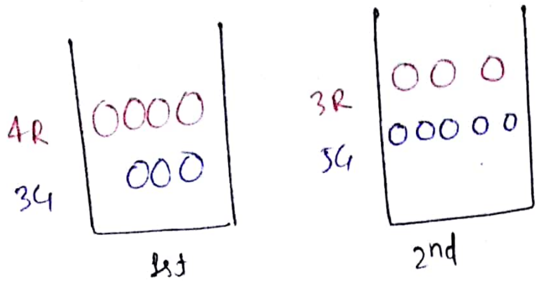
Exclusive event or mutually exclusive

To events A and B are said to be exclusive if

$P(A \cap B) = 0$

i.e Happening of one event prevents the happening of the other. then the two events are said to be exclusive or mutually exclusive.

Taking a ball from a bag :-



Q A ball is drawn at random from one of the bags

(1) what's prob^r that it is a red ball

(2) and is found to be a red ball what is the prob^r that it is obtained from IInd ball.

Sol^n - Let E_1 — Event of selection of 1st bag

E_2 — Event of selection of 2nd bag

A $\cancel{E_3}$ → " " " of Red ball

(1) $P(E_1) \times P(A/E_1) + P(E_2)\, P(A/E_2)$

→ chance of selection of Red ball from 1st bag

$= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{3}{8}$

For the IInd problem we require Bayes theorem

Bayes Theorem

$$\boxed{P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\; P(A/E_2)}{\sum P(E_i)\; P(A/E_i)}}$$

$$= \frac{\frac{1}{2} \times \frac{3}{8}}{\frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{4}{7}}$$

Problem

Machine 1	M 2	M 3
10%	20%	30%

E_1 = event of selection of 1st machine
E_2 = " " " " 2nd "
E_3 = " " " " 3rd "
A = " " " " defective chip.

Prob that defective chip is from E_3

$$P\left(\frac{E_3}{A}\right) = \frac{P(E_3)\,P(A/E_3)}{\sum_1^3 P(E_i)\,P(A/E_i)}$$

$$= \frac{\frac{1}{3} \times \frac{30}{100}}{\frac{1}{3} \times \frac{10}{100} + \frac{1}{3} \times \frac{20}{100} + \frac{1}{3} \times \frac{30}{100}}$$

$=$

Q
Work book

Prob of man to know answer $= \frac{2}{3}$

Prob of guessing correct ans $= \frac{1}{4}$

PO 'good'
Q

$\frac{1}{5}$ $\frac{1}{5}$

Prob to missing parcel at P.O. 1

A parcel is send by a sender and is not received by the receiver
what is the prob^r that it is missing in P.O. 2

Sol^n - Let $E_1 \rightarrow$ Event of parcel to reach P.O 1

E_2 — " " " " P.O.2

$A \rightarrow$ " " " to missing.

Parcel missed - find the prob' that it is missed in PO2.

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\; P(A/E_2)}{\sum P(E_i)\; P(A/E_i)}$$

$$P(E_2/A) = \frac{\frac{4}{5} \times \frac{1}{5}}{1 \times \frac{1}{5} + \frac{4}{5} \times \frac{1}{5}}$$

Ask?

P. Office 2 पे parcel पहुँचने की probability

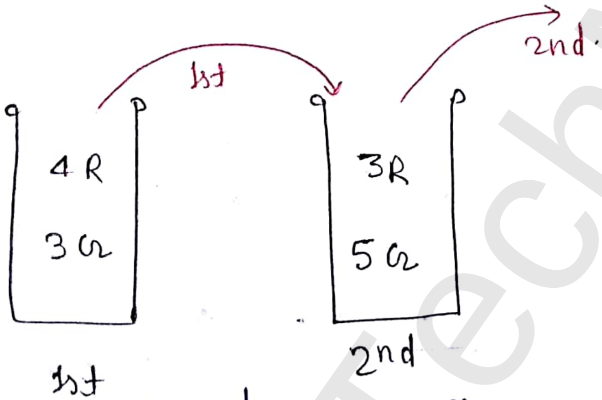
P. Office 2 पे parcel miss होने की probability.

Postoffice 1 पर parcel पहुँचने की probability

Post office 1 पर parcel miss होने की probability

Q

Find probability of getting red ball at 2nd draw from 2nd.



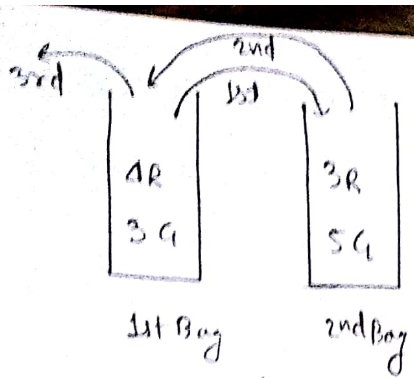
1st time Red, 2nd time red; 1st time green, 2nd time red

$R_1 R_2 + G_1 R_2$

$$P(R_1)\, P\left(\frac{R_2}{R_1}\right) + P(G_1)\, P\left(\frac{R_2}{G_1}\right)$$

$$\left(\frac{4}{7} \times \frac{4}{9}\right) + \left(\frac{3}{7} \times \frac{3}{9}\right)$$

Q



Prob of 3rd ball to be Red .?

$R_1R_2R_3 + R_1G_2R_3 + G_1R_2R_3 + G_1G_2R_3$

$\left(\frac{4}{7}\times\frac{4}{9}\times\frac{4}{7}\right)+\left(\frac{4}{7}\times\frac{5}{9}\times\frac{3}{7}\right)+\left(\frac{3}{7}\times\frac{3}{9}\times\frac{5}{7}\right)+\left(\frac{3}{7}\times\frac{6}{9}\times\frac{4}{7}\right)$

Atleast one to happen:

Q A room has 3 bulb holders and bag contains 15 bulbs of which 5 are fused, 3 bulbs are selected at random to fit into these bulbs holders, what is the prob^n that room gets lighted.

Sol^n prob for atleast one bulb to glow = 1 - prob of none bulb should glow

$$= 1 - \left(\frac{{}^5C_3}{{}^{15}C_3}\right)$$

Q The prob A to solve a problem in maths is $\frac{1}{3}$

" " B to " " " " " " $1/5$

" " C " " " " " " " $1/7$

what is the prob for the problem to be solved.

Sol^n P(atleast one to solve) = 1 - P(none of them solve)

$$= 1 - \left[\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\right]$$

Q The Prob for a man to hit a target is $\frac{1}{3}$ what is the prob for him to hit the target at least once in 5 chances.

Soln $P(hit) = \frac{1}{3}$ $P(miss) = \frac{2}{3}$

Prob (atleast one to hit) $= 1 - \left[\frac{2}{3}\right]^5$

Imp

Prob (atleast two times to hit) $= 1 - [P(\text{none of the times to hit}) + P(\text{one time to hit})]$

$$= 1 - \left[\left(\frac{2}{3}\right)^5 + {}^5C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4\right]$$

$nc_r\, p^r\, q^{n-r}$

Q The probability for a leap year selected at random to have

(1) 53 sundays is

(2) 52 sundays is

Soln 366 days

$7 \overline{)366}$ (52 — full week
35
16
14
2 ← 2 more days

53	Sun	sat	sun mon
52	mon	sat	S M T
52	T	sat	S M T W
52	W	sat	S M T W T
52	T	sat	S M T W T F
52	F	sat	S M T W T F S
53		sat	S M T W T F sat sun

2 days can be
SS ✓
SM ✓
MT
TW
WT
TF
FS

1) $\frac{2}{7}$

2) $\frac{5}{7}$

Q The same qusⁿ as above for non-leap year

365 days

$$7\overline{)365}\,52$$
$$\underline{364}$$
$$1$$

Sat
Sun ✓ 1) $\frac{1}{7}$
M
T
W
T 2) $\frac{6}{7}$
F
S

Statistics

Collection of data

Ist player   IInd

Play 3matches → "

1st player   IInd player

50 49 51   100 0 50

avg = 50   avg = 50

1st player is more consistant

Let we have ~~Ten~~ different data

$x_1 \quad x_2 \quad \ldots\ldots x_n$

Measures of Central Tendency

$\boxed{1. \text{ mean}}$
- AM $= \frac{\sum x_i}{n}$
- GM $= (\Pi x_i)^{1/n}$
- HM $= \frac{n}{\sum \frac{1}{x_i}}$

$\boxed{2. \text{ Median}}$
- middle most value of the data

$\boxed{3. \text{ mode}}$ – most frequently occuring value
* can be two values

measures of dispersion

1. Range = LV – S.V of the data (Largest value, smallest value)

2. mean deviation $= \frac{\sum |x_i - \bar{x}|}{n}$ ($\bar{x}$ → mean)

3. Standard deviation $= \sigma = \sqrt{\frac{\sum (x_i - x_n)^2}{n}} \geq 0$

4. Variance $= (S.D)^2 = \sigma^2$

$= \frac{\sum (x_i - \bar{x})^2}{n}$

arithmetic mean

Q The Am of 5, 10, 15, . . . 95 is

$S_n = \frac{5+95}{2} = 50$

when the terms are in AP $\boxed{Am = \frac{\text{1st term} + \text{last term}}{2}}$

Q The A.M of sum of no's going to form by rolling 4 dice is

4, 5, 6 24

$A.M = \frac{4+24}{2} = 14$

Q The mean deviation of $x_1, x_2 \dots x_n$ is 5 then M.D of $2x_1+3, 2x_2+3, \dots 2x_n+3$ is ____

a) 5 b) 10 c) 13 d) 20

Soln $x_1 \; x_2 \; x_3 \dots x_n$ } Initial data

$2x_1+3 \; 2x_2+3 \dots 2x_n+3$ } Resultant data

$AM_I = \frac{\sum x_i}{n} = \bar{x}$ (say)

$M.D_I = \frac{\sum |x_i - \bar{x}|}{n} = 5$ given

$A.M_R = \frac{2\sum x_i}{n} + \frac{3n}{n} = 2\bar{x} + 3$

$M.D_R = \frac{\sum |(2x_i+\cancel{3}) - (2\bar{x}+\cancel{3})|}{n} = \frac{2\sum |x_i - \bar{x}|}{n} = 2 \times 5 = 10$

Q The Variance of $x_1, x_2 \ldots x_n$ is 5 then the variance of $2x_1+3, 2x_2+3, \ldots 2x_n+3$ is ____.

a) 5 b) 10 c) 13 d) 20

Soln

$$A.M_I = \frac{\Sigma x_i}{n} = \bar{x} \text{ (say)}$$

$$\text{Variance}_I = \frac{\Sigma [x_i - \bar{x}]^2}{n} = 5 \text{ given}$$

$$A.M_R = \frac{2\Sigma x_i}{n} + \frac{3n}{n} = 2\bar{x}+3$$

$$\text{Variance}_R = \frac{\Sigma [(2x_i+3) - (2\bar{x}+3)]^2}{n}$$

$$= \frac{2^2 \Sigma (x_i - \bar{x})^2}{n} = 2^2 \times 5 = 20$$

Note* 1. By adding a fixed constt to each data point the dispersion value will not change

2. By multiplying a fixed constt to each data point range, mean deviation, Standr Deviation are multiplied by the constt whereas the variance is multiplied by square of the constant. i.e

$$\boxed{V(ax+b) = a^2 V(x)}$$

$$\boxed{\sigma(ax+b) = a\sigma(x)}$$

$$\boxed{E(ax+b) = aE(x)+b}$$

Random Variable :

When we toss a coin 3 times

8 case

no. of heads

TTT → 1

$\left.\begin{matrix} TTH \\ THT \\ HTT \end{matrix}\right\} 3$

$\left.\begin{matrix} HHT \\ HTH \\ THH \end{matrix}\right\} \to 3$

HHH } → 1

x	0	1	2	3
$P(x = x_i)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\sum P(x_i) = 1$$

Discrete RV

Continious RV

Prob distribution func $P(x)$

$\cancel{P(x)} = \sum P_i = 1$

Prob. density func or $f(x)$

prob mass func

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$P(a < x < b) = \int_a^b f(x)\,dx$$

	Mean	Variance
Discrete R.V	$\mu = E(x) = \sum x_i p_i$	$\sigma^2 = V(x) = E(x^2) - (E(x))^2 = \sum x_i^2 p_i - \mu^2$
Continious R.V	$\mu = E(x) = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx$	$\sigma^2 = V(x) = E(x^2) - (E(x))^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,dx - \mu^2$

<u>properties</u> :-

(1) $E(ax+b) = aE(x) + b$

(2) $V(ax+b) = a^2 V(x)$

easy

Q A r.v x has the following distribution

x	1	2	3	4	5	6	7
$P(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

then find 1) k 2) $P(1<x<4)$ 3) $P(x>6)$

we have $\sum p_i = 1$

$$10k^2 + 9k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k-1)(k+1) = 0$$

$$k = \frac{1}{10} \qquad k = -1$$

2) $P(1<x<4)$

$P(2) + P(3)$

$$= 2 \times \frac{1}{10} + 2 \times \frac{1}{10} = \frac{4}{10}$$

3) $P(x>6) = P(x=7)$

$$= \frac{7}{100} + \frac{1}{10} = \frac{17}{100}$$

Ex: Q A R[redacted]posses the following density fun^c

$$f(x) = k(1-x^2) \quad \text{for } 0<x<1$$
$$= 0 \quad \text{otherwise}$$

then find 1) k 2) μ

Soln We have $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} 0 + \int_{0}^{1} k(1-x^2)\,dx + \int_{1}^{\infty} 0 = 1$$

$$k = \frac{3}{2}$$

2) $\mu = \int_0^1 x \, \frac{3}{2}(1-x^2)\,dx$

$$= \frac{3}{2}\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1\right]$$

$$= \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{3}{8}$$

Q A raincoat dealer gets a profit of Rs. 200 /day if it rains and a loss of Rs. 20 if it is not raining the prob for a rainy day is 0.3 then what is the expectation of his business.

Soln

x → ~~Rupee~~ profit/day	200	-20
$P(x=x_i)$	0.3	0.7

$$E(x) = \Sigma_i x_i p_i = 200 \times 0.3 - 20 \times 0.7$$
$$= 46$$

Binomial Distribution func

$$P(x=r) = {}^nC_r \, p^r q^{n-r}$$

n = no of times Exp conducted

p = Prob of success each time

q = Prob of failure each time $= (1-p)$

mean $= np$

var $= npq$

Ex: Tossing a coin 3 times

$x \longrightarrow$ no. of heads turn up

here $n = 3$ $\quad p = \frac{1}{2}$ $\quad q = \frac{1}{2}$

$$P(x=0) = {}^3C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{3-0} = \tfrac{1}{8}$$

$$P(x=1) = {}^3C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^{3-1} = \tfrac{3}{8}$$

x	0	1	2	3
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Mean $= \sum x_i P_i = 0 + \frac{3}{8} + 6/8 + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$

using B.D mean $= n_i P_i = 3 \times \frac{1}{2} = 3/2$

similarly variance $= npq = 3/4$

Poisson distribution:

$$P(x=r) = \frac{e^{-\lambda}\lambda^{r}}{r!} \quad \text{or} \quad \frac{e^{-m}m^{r}}{r!}$$

Note: 1) In general poisson distribution is used when the no. of experiment conducted 'n' are very large and prob. of success 'p' is very small

2) To calculate λ we take $\lambda = np$

3) For the poisson distribution mean = Variance = λ

Poisson is limiting case of Binomial Distribution.

ex. $p = 0.01$

$n = 500$

$r = 2$

using BD

$$P(x=2) = {}^{500}C_{2}\,(0.01)^{2}(0.99)^{498}$$

using PD

$$P(x=2) = \frac{e^{-5}5^{2}}{2!} =$$

Normal Distribution or Gaussian Distribution.

Its density fun^{c} is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

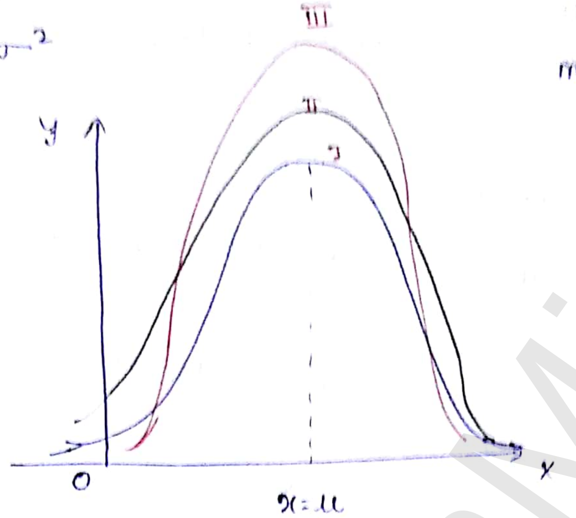
Mean = μ

Variance = σ^2

Stnd deviation = S.D

$SD_1 > SD_2 > SD_3$

more peak means less S.D



$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

If f(x) is non-integrable func

To calculate $P(a<x<b)$ we need to evaluate an Integral of the form

$$\int_{a}^{b} k e^{-\alpha x^2}\,dx$$

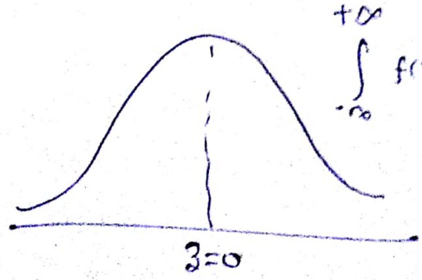
type of integral which is not possible analytically hence we convert the normal distribution to standard normal distribution.

S.N.D

By taking $z = \dfrac{x-\mu}{\sigma}$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2)}$$

$$\int_{-\infty}^{+\infty} f(z)\,dz = 1$$



mean = 0

var = 1

Benefit is that tabular values are provided for $P(0<z<z_i)$ for different z_i's.

Q The avg height of a student in a class is 175 cm with a standard Deviation of 10 cm a student is selected at random from that class, what is the prob for him to have

1) height b/w 170 cm to 180 cm

2) more than 180 cm given ~~prob of~~ $P(0<z<0.5)$ is .1915

Soln x is random variable

$x \rightarrow$ height of the student

has $\mu = 175$ $\sigma = 10$

(1) $P(170<x<180)$

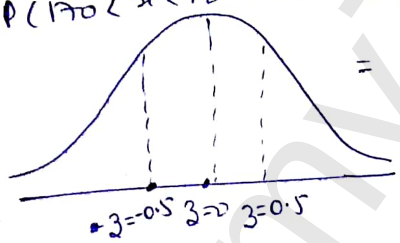
*** $\boxed{\text{we can't } \int e^{-\alpha x^2} \text{ so use S.N.D}}$

Let $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-175}{10}$

when $x = 170$ $z = -0.5$

when $x = 180$ $z = 0.5$

(1) $P(170<x<180) = P(-0.5<z<0.5)$



$= 2P(0<z<0.5)$

$= 2 \times .1915$

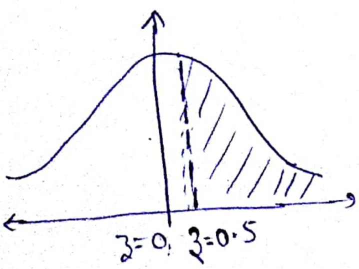
$= .383$

2) $P(x>180) = P(z>0.5)$

$= 0.5 - P(0<z<0.5))$

$= 0.5 - 0.1915$

$= 0.3085$



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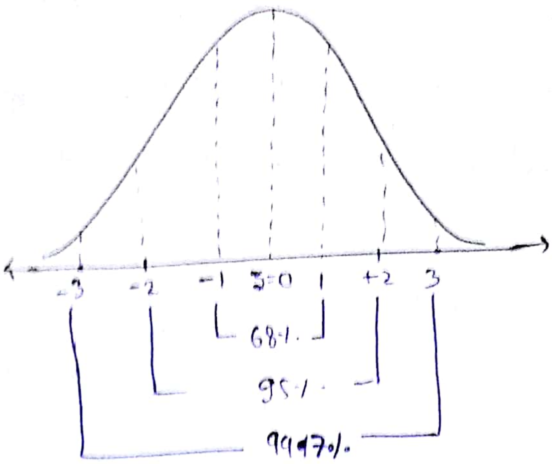
Standard Normal distribution curve



Exponential distribution

Its density fun^c is given by

$$f(x) = \theta e^{-\theta x} \quad \text{for } x \geq 0$$
$$= 0 \quad \text{for } x < 0$$

$$\text{Mean} = \frac{1}{\theta}$$

$$\text{Variance} = \frac{1}{\theta^2}$$

Q If $f(x) = 2e^{-2x}$ for $x \geq 0$

$= 0$ for $x < 0$

then $P(x > 0.5) =$

Sol^n $P(x > 0.5) = \int_{0.5}^{\infty} f(x)\, dx$

$$\int_{0.5}^{\infty} 2e^{-2x}\, dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = -1\left[0 - e^{-1}\right] = \frac{1}{e}$$

Uniform Distribution :

Its density funⁿ is given by $f(x) = \frac{1}{b-a}$ for $a < x < b$

$= 0$ otherwise

Mean $= \frac{a+b}{2}$

Variance $= \frac{(b-a)^2}{12}$

Q For a Uniform r.v x in the interval 2 to 5 the value of $P(1 < x < 3) =$ ___

$f(x) = \frac{1}{5-2}$ for $2 < x < 5$

$= 0$ otherwise

$$P(1 < x < 3) = \int_1^3 f(x)\,dx$$

$$= \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$0 + \int_2^3 \frac{1}{3}\,dx$$

$$= \frac{1}{3}$$

CALCULUS

① Differential

① Limits, continuity

② Diff, Roll's theorem

LMVT, ③ Partial diff.

④ Max and min, ⑤ tangent and Normal

② Integral

① definite integral

② sum of series with definite integral

③ Improper integral

④ multiple Integral

⑤ β and γ func

Calculus

Limit of a fun

↓

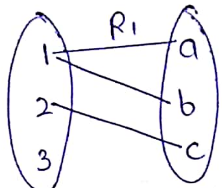
fun^c means $A \longrightarrow B$

↓

Set

$N = \{1, 2, 3 \ldots\}$ natural no. set · Discrete sets

Dense set



Not a fun^c from A to B

1) Every element of set A ~~should relate~~ must be associated

2) One element should not have two images.

(1, 2) (3, 4)

↳ this set contains every real no. b/w 1 & 2 excluding 1 and 2

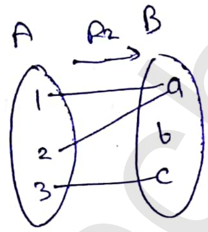
$A \xrightarrow{fun^c} B$

Discret —— Discrete

$A \xrightarrow{fun} B$

Dense —— Dense

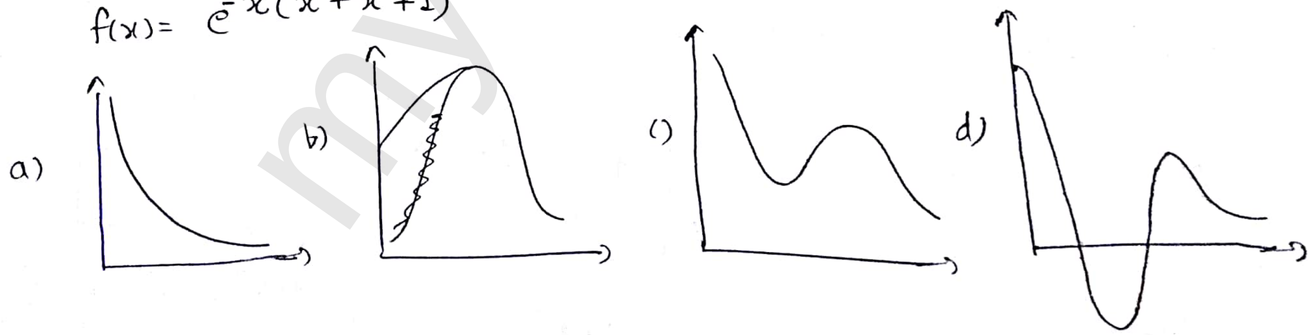
Discre $\xrightarrow{\text{Not } fun^c}$ Dense



this is a fun^c

Q) Which one of the following graph describes the fun^c

$$f(x) = e^{-x}(x^2 + x + 1)$$



option (b) using the cond^n of maxima minima

If $g(x) = 1-x$ $\quad h(x) = \frac{x}{x-1}$

then $\frac{g(h(x))}{h(g(x))} =$

a) $\frac{h(x)}{g(x)}$ $\quad$ b) $\frac{g(x)}{h(x)}$

c) $-\frac{1}{x}$ $\quad$ d) $\frac{x}{(1-x)^2}$

$$\frac{g\left(\frac{x}{x-1}\right)}{h(1-x)} = \frac{1-\frac{x}{x-1}}{\frac{1-x}{1-x-1}} = \frac{\frac{-1}{x-1}}{\frac{1-x}{-x}} = \frac{-x}{(1-x)^2}$$

Q Let $S = \sum_{n=0}^{\infty} n\alpha^n$ where $|\alpha|<1$ then value of α in the range $0<\alpha<1$ such that $S = 2\alpha$ is ______

$$S = 0+\alpha^1 + 2\alpha^2 + 3\alpha^3 + \cdots$$

$$S = \alpha[1+2\alpha+3\alpha^2+\cdots]$$

Remb^r

$$\boxed{\frac{1}{(1-\alpha)^2} = 1+2\alpha+3\alpha^2+\cdots}$$

given

$$2\alpha = \frac{\alpha}{(1-\alpha)^2}$$

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots$$

Binomial expansion.

$$(1-\alpha)^2 = \frac{1}{2}$$

$$1-\alpha = \pm\frac{1}{\sqrt{2}}$$

$$1\mp\frac{1}{\sqrt{2}} = \alpha$$

$$\alpha = 1-\frac{1}{\sqrt{2}} \qquad \alpha = 1+\frac{1}{\sqrt{2}}$$

$$\alpha = 0.2928$$

Q The value of $\lim_{n\to\infty}\left[\sqrt{n^2+n} - \sqrt{n^2+1}\right] =$

0 the value

a)

Solⁿ

$$\cancel{\lim_{n\to\infty}\sqrt{n^2+n} - \lim_{n\to\infty}\sqrt{n^2+1}}$$

solve this ~~complete this~~ as

$$\lim_{n\to\infty} \frac{(n^2+n) - (n^2+1)}{\sqrt{n^2+n} + \sqrt{n^2+1}}$$

$$\mathrm{Lt}_{n\to\infty} \frac{\cancel{n}\left[1-\frac{1}{n}\right]}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n^2}}\right)}$$

$$= \frac{1-0}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$$

$\ln(a-b)$

$\ln\left(\frac{a^2-b^2}{a+b}\right)$

$\ln\left(\frac{\cancel{(a+b)}(a-b)}{\cancel{(a+b)}}\right)$

$\boxed{\ln(a-b)}$

Q Let $g(x) = \begin{cases} -x & x \le 1 \\ x+1 & x \ge 1 \end{cases}$

$f(x) = \begin{cases} 1-x & x \le 0 \\ x^2 & x > 0 \end{cases}$

The no. of discontinuity into ~~f(ng(x))~~ $f(g(x))$ in the interval $(-\infty, 0)$

a) 0

b) 1

c) 2

d) 4

Solⁿ $f(g(x)) = f(-x)$

$= x^2$

option (a) ✓

DNK?

Q the value of $\int e^{-x - e^{-x}} dx =$

a) $e^{e^{-x}}$

b) $e^{-e^{-x}}$

c) $e^{-e^{x}}$

d) e^{-x}

Soln

$$\int e^{-x} \cdot e^{-e^{-x}} dx$$

put $e^{-x} = t$

$$\bar{e}^{x} dx = dt$$

$$= \int e^{-t} (-dt)$$

$$\bar{e}^{t} = e^{-e^{-x}}$$

Q $\underset{x \to 0}{Lt} \frac{\tan x}{x^2 - x} =$

Soln $\frac{0}{0}$ form

$$\underset{x \to 0}{Lt} \frac{\sec^2 x}{2x - 1} = \frac{1}{-1} = -1$$

Q If $f(x) = R\sin\left(\frac{\pi x}{2}\right) + S$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$

$\int_0^1 f(x)\,dx = \frac{2R}{\pi}$ then R and S respectively are

a) $\frac{2}{\pi}$ & $\frac{16}{\pi}$

b) $\frac{2}{\pi}$ & 0

c) $\frac{4}{\pi}$ & 0

d) $\frac{4}{\pi}$ & $\frac{16}{\pi}$

Soln $f'(x) = \frac{\pi}{2} R \cos\left(\frac{\pi x}{2}\right)$

$f'\left(\frac{1}{2}\right) = \frac{R\pi}{2} \cos\left(\frac{\pi}{4}\right) = \sqrt{2}$

~~$\cos\pi$~~

$\frac{R\pi}{2} \cos \frac{1}{\sqrt{2}} = \sqrt{2}$

$\boxed{R = 4/\pi}$

$\int_0^1 R\sin\left(\frac{\pi x}{2}\right) + S$

$\int_0^1 \frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right) + S(1-0) = \frac{2R}{\pi}$

$-\frac{4}{\pi} \frac{\cos\left(\frac{x\pi}{2}\right)}{\pi/2}\Big|_0^1 + S = \frac{2}{\pi} \cdot \frac{4}{\pi}$

$-\left\{0 - \frac{4}{\pi}\cos 0\right\} + S = \frac{8}{\pi}$

$f'(x) = R \cos\frac{\pi x}{2} \cdot \frac{\pi}{2}$

$f'\left(\frac{1}{2}\right) = R \cdot \frac{\pi}{2} \cos\frac{\pi \cdot 1}{2 \cdot 2} = \sqrt{2}$

$\frac{R\pi}{2} \cos\frac{\pi}{4} = \sqrt{2}$

$\boxed{R = 4/\pi}$

$\int_0^1 f(x)\,dx = \frac{2R}{\pi}$

$\int_0^1 \left(R\sin\left(\frac{\pi x}{2}\right) + S\right) dx = \frac{2R}{\pi}$

$-R\cos\frac{\pi x}{2}\Big|_0^1 \cdot \frac{2}{\pi} + S = \frac{2R}{\pi}$

$-\left\{0 - \cos 0\right\} R \cdot \frac{2}{\pi} + S = \frac{2R}{\pi}$

$\frac{4}{\pi} \cdot \frac{2}{\pi} + S = \frac{2 \cdot 4}{\pi \cdot \pi}$

$\boxed{S = 0}$

(C)

Q A fun^c f(x) is defined as

$$f(x) = \begin{cases} e^x & x<1 \\ \ln x + ax^2 + bx & x \geq 1 \end{cases}$$ where $x \in R$

which is true

a) f(x) is not differentiable at $x=1$ for any values of a & b

b) f(x) is diff at $x=1$ for unique value of a & b

c) " " " " " " for all a & b for which $a+b=e$

d) f(x) is diff at $x=1$ for all a & b.

$$f'(x) = \begin{cases} e^x & x<1 \\ \frac{1}{x} + 2ax + b & x \geq 1 \end{cases}$$

$$f'(1^+) = f'(1^-)$$

$$1 + 2a + b = e$$

$$2a + b = e - 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\ln(1) + a + b = e$$

$$a + b = e$$

$$\frac{2a + b = e - 1}{-a = 1} \quad a = -1$$

$$b = 1 + e$$

(b) ✓

Q A 3 dimensional region R of finite volume is described by

$x^2 + y^2 \le z^2$, $0 \le z \le 1$ The volume of R correct to 2 decimal places is

$$V = \int_{z_1}^{z_2} \pi r^2 \, dz = \int_0^1 \pi z^3 \, dz$$

Ans. Now

$$= \pi \left(\frac{z^4}{4} \right)_0^1 = \frac{\pi}{4}$$

Q the min^m value of func $f(x) = \frac{1}{3} x(x^2 - 3)$ in the interval $-100 \le x \le 100$ occurs at $x =$

$$f(x) = \frac{x^3}{3} - x$$

$$f'(x) = \frac{3x^2}{3} - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

at $f(1) = \frac{1}{3}(1-3) = \frac{-2}{3}$

$f(-1) = \frac{-1}{3}(1-3) = \frac{2}{3}$

$f(-100) = \frac{-100}{3}(100^2 - 3) = -(1000-3) \times \frac{100}{3} = \approx \frac{-100000}{3}$

$\boxed{x = -100}$ Ans

Q $\lim_{x \to 0} \dfrac{e^{2x}-1}{\sin 4x}$

$\lim_{x \to 0} \dfrac{0}{0}$

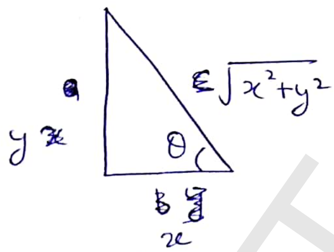
$\lim_{x \to 0} \dfrac{2e^{2x}}{4\cos 4x}$

$= \dfrac{2}{4} = 0.5$

Q For a right angled triangle if the sum of lengths of hypotenuous and a side is kept constant in order to have max^m ~~volume~~ area of the triangle then angle b/w the hypotenuse under the side is

a) 12° b) 36° c) 60° d) 45°

Soln



~~b+c = const~~

$x + \sqrt{x^2+y^2} = \text{const} = c$

$x^2+y^2 = (c-x)^2$

$y^2 = c^2 + x^2 - 2cx - x^2$

$y^2 = c^2 - 2cx$

Area $= A = \frac{1}{2}xy = \frac{1}{2}$

$A^2 = \frac{1}{4}x^2y^2$

$A^2 = \frac{1}{4}x^2(c^2 - 2cx) = f(x)$] (say)

$$f'(x) = \frac{1}{4}\left[c^2 2x - 2c(3x^2)\right] = 0 \qquad x = 0, \frac{c}{3}$$

$$f''(x) = \frac{1}{4}\left[c^2 2 - 12cx\right]$$

$$f''(0) = \frac{1}{4} 2c^2 > 0 \qquad \text{(minima)}$$

$$f''\left(\frac{c}{3}\right) = \frac{1}{4}\left[2c^2 - \frac{12c^2}{3}\right]$$

$$= \frac{1}{4}\left[2c^2 - 4c^2\right]$$

$$f''\left(\frac{c}{3}\right) = \frac{-2c^2}{4} = \frac{-c^2}{2} < 0 \qquad \text{(maxima)}.$$

Maxima at $x = \frac{c}{3}$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{\frac{c}{3}}{\left(c - \frac{c}{3}\right)} = \frac{\frac{c}{3}}{\frac{2c}{3}} = \frac{1}{2}.$$

$\cos\theta = \frac{1}{2}$

$$\boxed{\theta = 45^\circ}$$

---

if i find $\tan\theta = \frac{y}{x} = \frac{\sqrt{c^2 - 2cx}}{x} = \frac{\sqrt{c^2 - 2\cdot c\cdot \frac{c}{3}}}{\frac{c}{3}}$

$$\tan\theta = \frac{\sqrt{c^2 - \frac{2c^2}{3}}}{\frac{c}{3}} = \frac{c\sqrt{1-\frac{2}{3}}}{\frac{c}{3}} = \frac{\frac{c}{\sqrt{3}}}{\frac{c}{3}} = \sqrt{3}$$

$\tan\theta = \sqrt{3}$

$\theta = 60^\circ$ why?

Q The eq^n of tangent to the curve $\sqrt{x} + \sqrt{y} = 5$ at $(9,4)$ is ____

a) $x+y=13$
b) $x+2y=17$
c) $2x+3y=27$
d) $2x+3y=30$

to get tangent
Sol^n diff wrt x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)_{(9,4)} = -\frac{2}{3}$$

eq^n of tangent is $\qquad (y-y') = \text{slope } (x-x')$

$$y-4 = -\frac{2}{3}(x-9)$$

(d) ✓

Q The derivative of $\ln \sec x$ w.r.t. $\tan x$ is

a) $\sin x$
b) $\cos x$
c) $\tan x$
d) $\sin \cos x$

Sol^n $\dfrac{d \ln \sec x}{d \tan x}$

let $z = \ln \sec x \qquad y = \tan x$

$$\frac{dz}{dy} = \frac{\frac{dz}{dx}}{\frac{dy}{dx}} = \frac{\frac{1}{\sec x}\sec x \cdot \tan x}{\sec^2 x} = \sin x \cdot \cos x$$

Q The number of distinct extreme values of $f(x) = 3x^4 - 16x^3 + 24x^2 + 37$ is ________

a) 0

b) 1

c) 2

d) 3

Extreme ~~value~~ value means it should be either maxima or minima but not a saddle point

$f'(x) = 12x^3 - 48x^2 + 48x = 0$

$x(x^2 - 4x + 4) = 0$

$x[x-2]^2 = 0$ $\qquad \underline{x = 0, 2, 2}$

$\downarrow$

these are stationary point

these may or may not be extreme point

$f''(x) = 12\{3x^2 - 8x + 4)$

$f''(0) = 48 > 0$ minima at $x = 0$

$f''(2) = 0$ saddle point

So minima at one point i.e $x = 0$ so (b) ✓

Q If $y = x+\sqrt{x+\sqrt{x+\ldots\infty}}$ then $y(2) =$ ____

a) 4 or 1

b) 4 only

c) 1 only

d) undefined

Soln:-

$\cancel{y = x +}$

$y - x = \sqrt{y}$

$(y-x)^2 = y$

$y^2 - 2xy + x^2 = y$

$y^2 - 4y + 4 = y$

$y^2 - 5y + 4 = 0$

$(y^2 - 4y) - (y \mp 4) = 0$

$y(y-4) - 1(y-4) = 0$

$y = 1,\ y = 4$

$y(2) = 2 + (\quad)$

$\therefore\ y(2) = 4$ only

Q. If at every pt on a curve slope is $-\frac{2x}{y}$ then the curve is ____

a) St. line

b) parabola

c) ~~elli~~ circle

d) ellipse.

$$\frac{dy}{dx} = \frac{-2x}{y}$$

$$y\,dy + 2x\,dx = 0$$

$$\frac{y^2}{2} + x^2 = c$$

$$\frac{x^2}{c} + \frac{y^2}{2c} = 1$$

ellipse not a circle bcz x^2 y^2 cofficient are not same.

Q The maximum value of $f(x) = x^2 - x - 2$ $[-4,4]$ is

a) 18

b) 10

c) -2.25

d) indeterminant.

$f'(x) = 2x - 1 = 0$ $\qquad x = 1/2$

$f''(x) = 2$

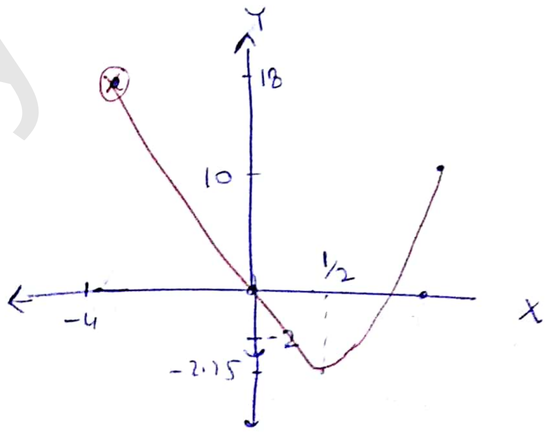
$f''(1/2) = 2 > 0 \Rightarrow$ minima at $x = \frac{1}{2}$

$f(-4) = 18$

$f(0) = -2$

$f(1/2) = -2.25$

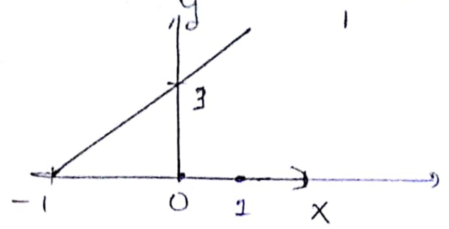
$f(4) = 10$



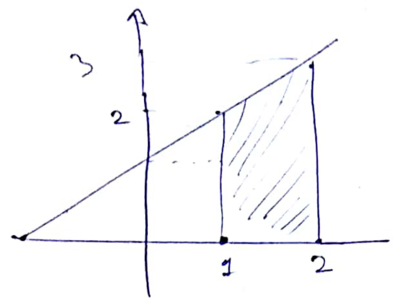
if in qusn the interval $[-4\ 4]$ if is not given.

then maximum value of $f(x)$ = indeterminant.

Q The following plot shows a fqn^c "y" which varies linearly with 'x' then the values of $\int_1^2 y\,dx$ is ____



Soln



$$\int_1^2 (1+x)\,dx$$

$$= (x)_1^2 + \left(\frac{x^2}{2}\right)_1^2$$

$$= (2-1) + \left(\frac{4}{2} - \frac{1}{2}\right)$$

$$= 1 + 2 - \frac{1}{2}$$

$$= 2.5$$

Q If $u = \ln\left(\frac{x^4+y^4}{x-y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$

a) 4
b) 3u
c) 3
d) e^u

Soln If z is not a homogeneous func but $f(z)$ is a homogeneous func of degree n then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\frac{f(z)}{f'(z)} = G(z)$ say

2) $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = G(z)[G'(z)-1]$

(c) ✓

Q $z = \ln(x^3+y^3-x^2y-xy^2)$ then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} =$

a) $\frac{1}{x+y}$

b) $\frac{2}{x+y}$

c) $\frac{y}{x+y}$

d) $\frac{x}{x+y}$

Soln. $f(z) = x^3+y^3-x^2y-xy^2$

$u = f(z) =$ homoge. func of order 3

By Sir

$$\frac{\partial z}{\partial x} = \frac{1}{\cdots\cdots}(3x^2-2xy-y^2) + \frac{1}{\cdots\cdots}(3y^2-x^2-2xy)$$

$$= \frac{2x^2+2y^2-4xy}{\cdots\cdots}$$

$$= \frac{2(x-y)^2}{(x^2-y^2)(x-y)} = \frac{2}{x+y}$$

Q In which interval $f(x) = (x-1)^2(x+1)^3$ is ↑ing

a) $(5, \infty)$

b) $(1, 5)$

c) $(1, \infty)$

d) (None)

Soln. at $x=1 = 0$

$x = \infty =$

By Sir $f'(x) = (x-1)^2(-3(x+1)^{-4}) + (x+1)^{-3}(2(x-1))$

$= (x-1)(x+1)^{-4}[-3(x-1) + 2(x+1)]$

$= (x-1)(x+1)^{-4}(5-x) > 0 \ \forall x\ (1,5)$

Q The maximum value of $f(x) = x^3 - 9x^2 + 24x - 12$ obtains at $x =$ ____

$f'(x) = 3x^2 - 18x + 24 = 0$

$x^2 - 6x + 8 = 0$

$\frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm 2}{2} = 4, 2$

$f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -ve \Rightarrow$ maxima at 2

Q The value of c of the MVT for $f(x) = (x-a)^m (x-b)^n$ in $[a,b]$ where $m, n \in N$ is ____

Soln a) $\frac{a+b}{2}$

b) $\frac{ab}{2}$

c) $\frac{mb+na}{2}$

d) $\frac{ma+nb}{m+n}$

Soln every polynomial funᶜ is differential through out Real line

$f'(x) = 0$

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$$m(x-a)^{m-1}(x-b)^{n-1} + (x-a)^{m}(n(x-b)^{n-1}) = 0$$

$$(x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)] = 0$$

$$m(x-b) + n(x-a) = 0$$

$$x = \frac{mb+na}{m+n} \in (a,b)$$

Q The values of $\int_0^2 |1-x|\,dx$

$x=1$

$$-\int_0^1 (1-x) + \int_1^2 (1-x)$$

$$(x)_0^1 - \left(\frac{x^2}{2}\right)_0^1 + (x)_1^2 - \left(\frac{x^2}{2}\right)_1^2$$

$$(1-0) - \left(\frac{1}{2}\right) + (2-1) - \left\{\frac{4}{2} - \frac{1}{2}\right\}$$

$$1 - \frac{1}{2} + 1 - \frac{3}{2}$$

$$2 - \left(\frac{1}{2} + \frac{3}{2}\right)$$

wrong $= 0$

By Sir

$|x| = x$ for $x \geq 0$

$= -x$ for $x \leq 0$

$|1-x| = 1-x$ for $1-x \geq 0 \Rightarrow x \leq 1$

$= -(1-x)$ for $1-x < 0 \Rightarrow x > 1$

$$= \int_0^1 (1-x)\,dx + \int_1^2 (1-x)\,dx$$

$$= (x)_0^1 - \left(\frac{x^2}{2}\right)_0^1 + \left[\left(\frac{x^2}{2}\right)_1^2 - (x)_1^2\right]$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Q $\int_0^1 \frac{\ln(1+x)}{(1+x^2)}\,dx$

a) $\frac{\pi}{4}\ln 2$

b) $\frac{\pi}{8}\ln 2$

c) $\frac{\pi}{4}\ln\sqrt{2}$

d) $\frac{\pi}{8}\ln\sqrt{2}$

put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

UL when $x=1$ $\theta = \pi/4$

LL when $x=0$ $\theta = 0$

$$\int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\cdot \sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = I$$

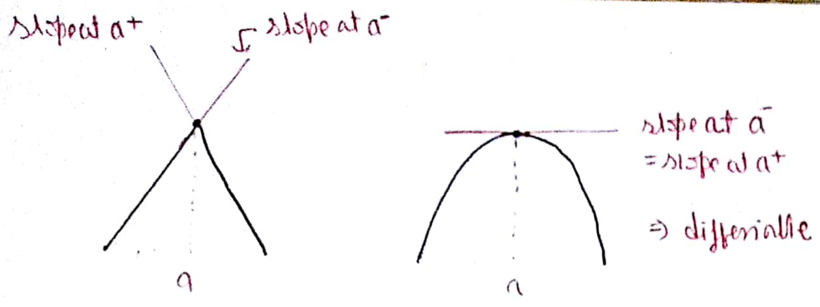
$$I = \int_0^{\pi/4} \ln(1+\tan(\pi/4-\theta))\, d\theta$$

$$= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \ln 2\, d\theta - I$$

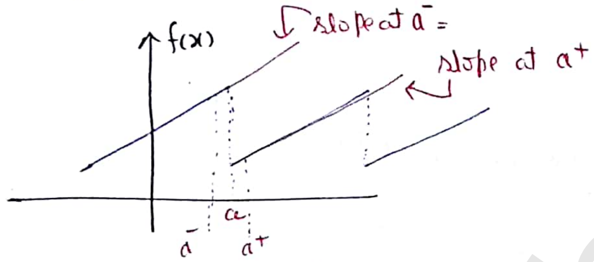
$2I = \ln 2\,\frac{\pi}{4} \Rightarrow I = \pi/8 \ln 2$



Not differentiable.

* To check the differentiability of a fun^c at a point, first we need to check the continuity of that function at that point.

Ex.



Is this function differentiable at point a.?

Sol^n If we don't check the continuity of fun' f(x). and directly jump to differentiability property.

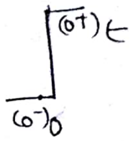
we will see that slope at a^- = slope at a^+ i.e

$$f'(a^-) = f'(a^+) \Rightarrow \text{Differentiable.}$$

but actually not differentiable. b/z the fun^c is not continious. that's the reason why First we check the continuity of the function.

For checking continuity

here $f(a^-) \neq f(a^+)$ so not continious.



Is this a continious func.

Date - 30 July

Multiple Integral

$$\boxed{\int_c^d \boxed{\int_a^b f(x,y)\, dx}\; dy}$$

1) Solve inner braket
2) $\int$ w.r.t x treat y as constt
(3) 1st Inner braket always should have variable limit.
(4)

Q. $$\int_{x=0}^{1}\int_{y=0}^{x} (x+y)\, dy\, dx$$

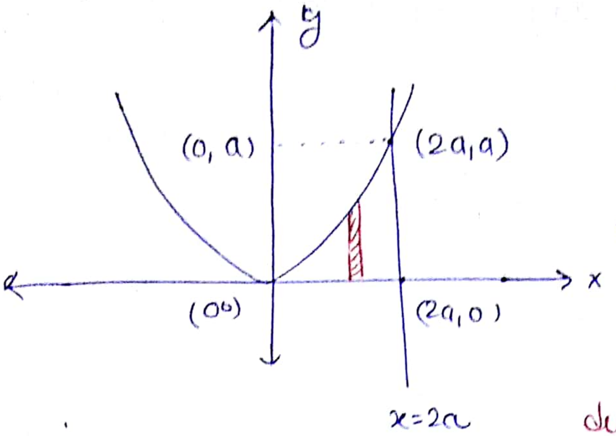
$$\int_0^1 \left(x\left[y\right]_0^{x} + \left(\frac{y^2}{2}\right)_0^{x} \right) dx$$

$$= \int_0^1 \left(x^2 + \frac{x^2}{2}\right) dx$$

$$\frac{3}{2}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2}$$

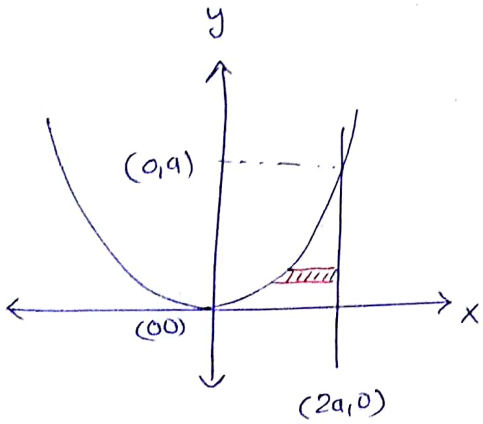
Q The value of $\iint_R xy\, dx\, dy$ where R is the reason bounded by x-axis $x = 2a$ & $x^2 = 4ay$

Soln



$$\int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx \quad -(1)$$

due to vertical strip inner box limit starts from y



$$\int_{y=0}^{a}\int_{x=\sqrt{4ay}}^{2a} xy\,dy\,dx \quad -(2)$$

Solving (1)

$$\int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx = \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_{x=0}^{2a} \frac{x}{2}\left[\frac{x^4}{16a^2}\right]dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{a^4}{3}$$

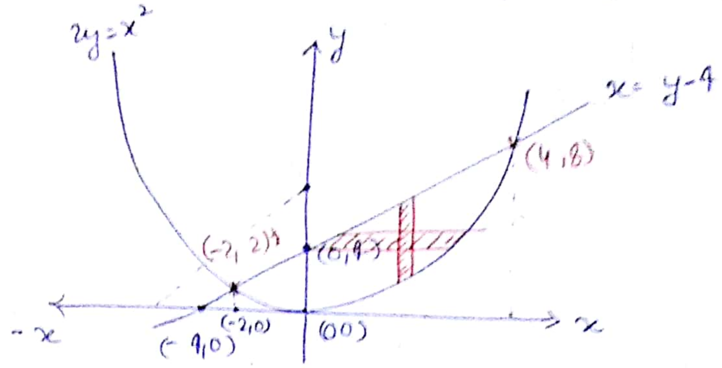
Solving (2) also gives $= \dfrac{a^4}{3}$

Note* 1) $\iint_R dx\,dy$ always represents the area bounded by the region "R".

IInd Model

Q Find the area bounded b/w ~~$x^2 = 4ay$~~ $2y = x^2$ and $x = y-4$

Sol^n



Required area $= \iint_R dy\,dx$

$$2y = (y-4)^2$$
$$2y = y^2 + 16 - 8y$$
$$y^2 - 10y + 16 = 0$$
$$\frac{10 \pm \sqrt{100-64}}{2} = \frac{10 \pm 6}{2} = \frac{16}{2}, \frac{4}{2}$$
$$y = 8, \ 2$$

$$\text{Area} = \int_{x=-2}^{x=4} \int_{y=x^2/2}^{y=x+4} dy\,dx$$

$$\int_{-2}^{4} \left(x + 4 - \frac{x^2}{2} \right) dx$$

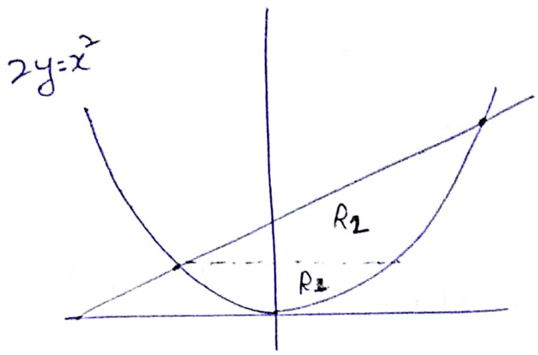
$$\left[\frac{x^2}{2} + 4x - \frac{x^3}{3} \right]_{-2}^{4}$$

18 in area

$$= \left(\frac{16}{2} + 16 - \frac{64}{3} \right) - \left(\frac{4}{2} - 8 + \frac{8}{3} \right)$$

~~$(8+16-21$~~

$$24 - \frac{64}{3} - 2 + 8 - \frac{8}{3} = 30 - \frac{72}{3} = 30 - 24 = 6$$



Req area $= \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$

$$= \int_{y=0}^{2} \int_{x=-\sqrt{2y}}^{\sqrt{2y}} dx\,dy + \int_{y=2}^{8} \int_{x=y-4}^{\sqrt{2y}} dx\,dy$$

we can solve like this

$$= \int_{-2}^{4} \int_{x^2/2}^{x+4} dy\,dx$$

$$= \int_{-2}^{4} \left(x+4 - \frac{x^2}{2}\right) dx$$

$$= \left. \frac{x^2}{2} + 4x - \frac{x^3}{2 \cdot 3} \right|_{-2}^{4}$$

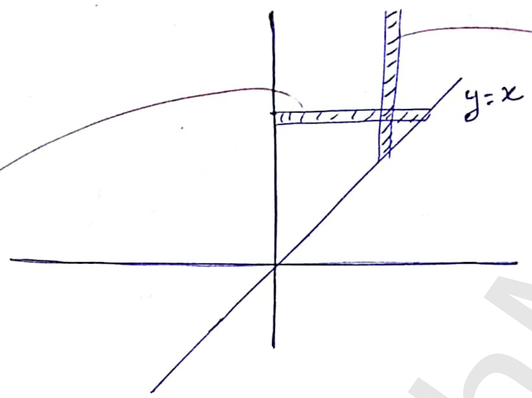
$$= \left(\frac{16}{2} + 16 - \frac{64}{6}\right) - \left(\frac{4}{2} - 8 + \frac{8}{6}\right)$$

$= 18$ //

III^rd Module

Q the value of $\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx$.

$\frac{e^{-y}}{y}$ is not integrable wrt y so we need to change the limits.



Before the question was in this form

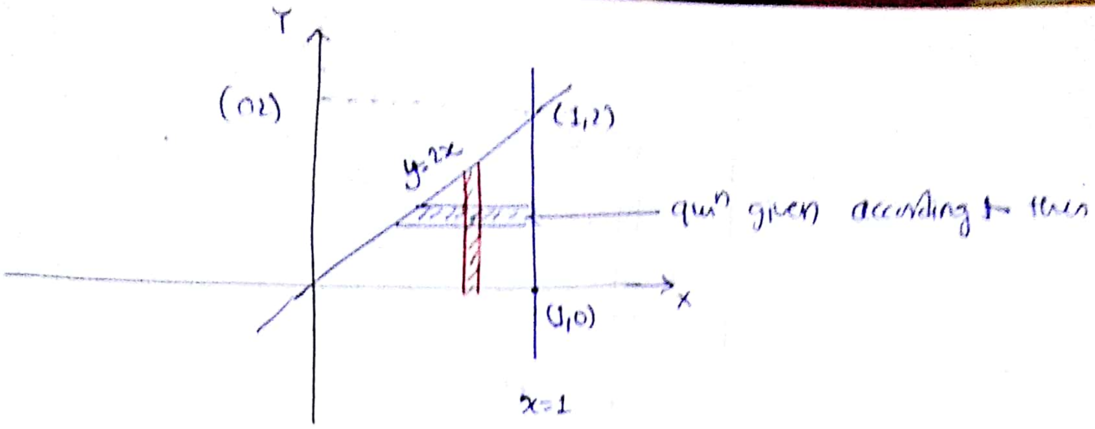
$\int_{x=0}^{\infty}\int_{x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx$

$$\int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}[x]_0^y\,dy = \left(-e^{-y}\right)_0^{\infty} = 1$$

Q The value of $\int_0^2\int_{y/2}^{1} \frac{1}{(1+x^2)^3}\,dx\,dy$.

$\frac{1}{(1+x^2)^3}$ is not directly $\int$ integrable.



$$= \int_{x=0}^{1} \int_{y=0}^{2x} \frac{1}{(1+x^2)^3}\, dy\, dx$$

Qu^n can be

$$\int_0^2 \int_{y/2}^{1} f(x,y)\, dx\, dy = \int_c^d \int_a^b f(x,y)\, dy\, dx$$

what is the values of $\underline{b}$?

$$= \int_{x=0}^{x=1} \frac{1}{(1+x^2)^3} \times 2x\, dx$$

put $1+x^2 = t \Rightarrow 2x\,dx = dt$

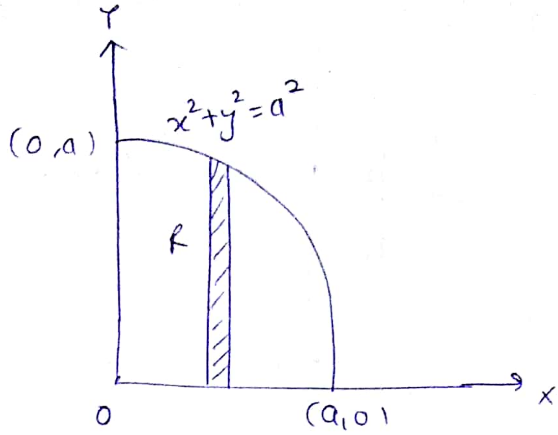
UL when $x=1$ $t=2$

L.L when $x=0$ $t=1$

$$= \int_1^2 \frac{1}{t^3}\, dt = \left[\frac{t^2}{2}\right]_1^2 = \frac{3}{8}$$

IV^th model. changing the limit don't work. so change fun^c in Polar coordinate.

(P) The value of $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx$



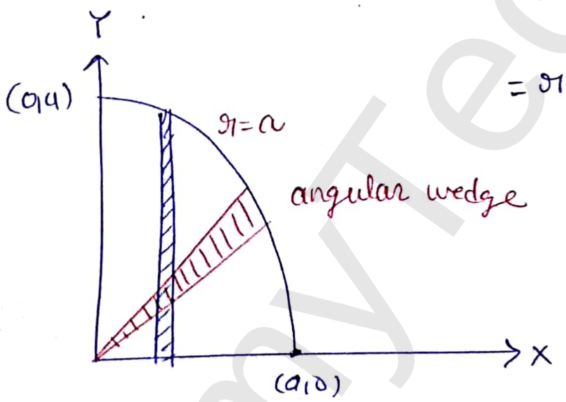
$x = r\cos\theta$, $y = r\sin\theta$ $\qquad dx\,dy = \phi(r,\theta)\,dr\,d\theta$

$= r\,dr\,d\theta$

$$\phi(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & \cancel{\cos\theta}\ r\cos\theta \end{vmatrix}$$

$= r$



$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a} e^{-r^2}\, r\, dr\, d\theta$$

$\int e^{-x^2} x\,dx$ $\qquad$ put $x^2 = t$

$x\,dx = \frac{dt}{2}$

$= \int e^{-t} \frac{dt}{2}$

$$\frac{1}{-2}e^{-t} = -\frac{1}{2}e^{-x^{x}}$$

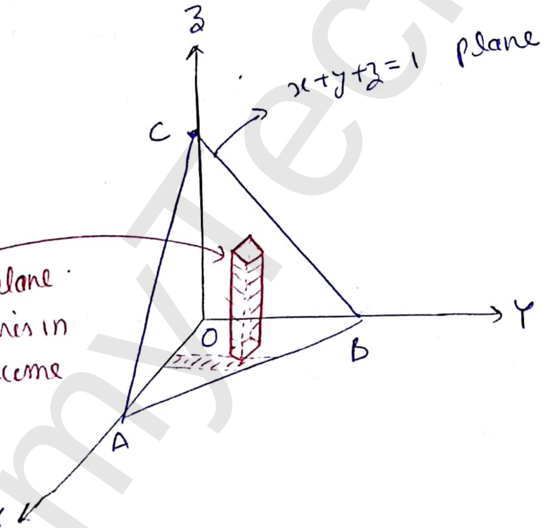
$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} e^{-r^2}\, r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left(\frac{-1}{2}e^{-r^2}\right)_0^a d\theta$$

$$= -\frac{1}{2}\left[e^{-a^2}-1\right] \times (\theta)_0^{\pi/2}$$

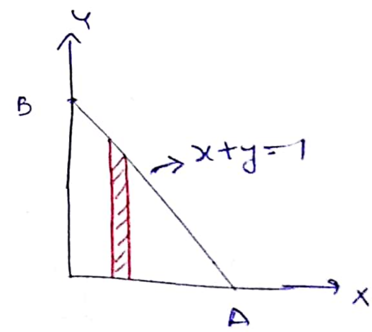
$$= \frac{\pi}{4}\left(1-e^{-a^2}\right)$$

Note* (2) $\iiint_V dx\,dy\,dz$ always represents the volume bounded by that V.

Q Find the volume bounded by the tetrahedron $x=0$, $y=0$ & $z=0$, $x+y+z=1$



i will have to move this in xy plane. if i will slide this in xy plane it will become complete volume.



Required volume $= \iiint dz\,dy\,dx$

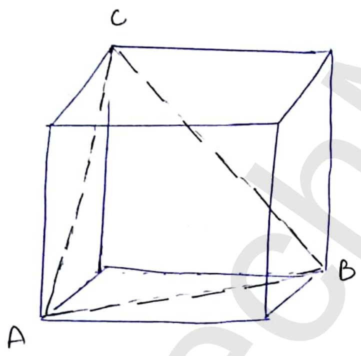
$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx$$

$$V = \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\, dy\, dx$$

$$\int_{x=0}^{1} \left((1-x)(y)_0^{1-x} - \left(\frac{y^2}{2}\right)_0^{1-x} \right) dx$$

$$V = \int_{x=0}^{1} \left[(1-x)^2 - \frac{(1-x)^2}{2} \right] dx = \frac{1}{2}\left[\frac{(1-x)^3}{-3} \right]_0^1$$

$$= -\frac{1}{6}[0-1] = \frac{1}{6} \text{ cubic unit}$$

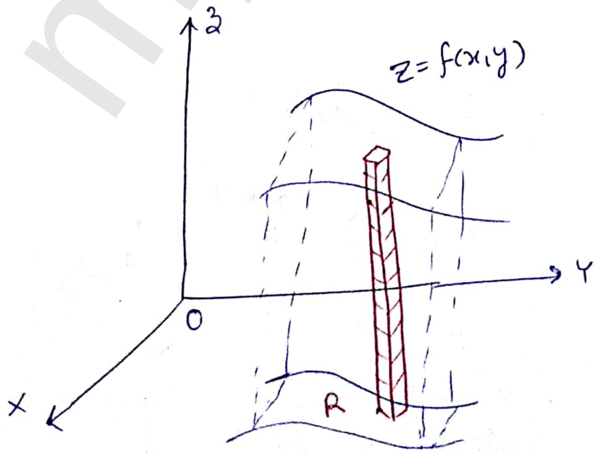


volume of cube $= 1 \times 1 \times 1 = 1$

tetrahedron covers $= \frac{1}{6}$

So if i am given 6 tetrahedron i will cover complete volume.

Note 3) $\iint_R f(x,y)\, dy\, dx$ represents volume ~~covered~~ bounded b/w the fun $z = f(x,y)$ which is a surface and xy plane.



remember Note(1), Note(2), Note(3)

Vector Calculus

Vector differential operator ∇ (del) :-

It is given by $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

1) $\text{grad}\,\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ → represents outward Normal to the surface $\phi = 0$

2) $\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial y}$ → If $\vec{F}$ denotes the force acting on a e's in electric field then $\text{div}\,\vec{F}$ denotes the amount of charge flowing per unit volume in unit time

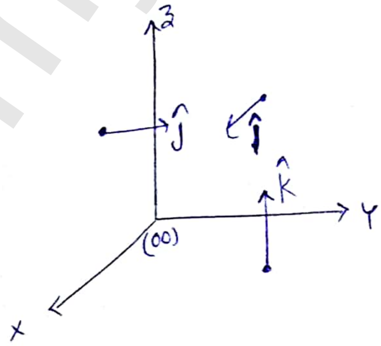
3) $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$ → If $\vec{F}$ denotes the Force acting on e's in electric field then curl denotes the 2 times the angular velocity of particle.

$\text{curl}\,\vec{F} = 2\omega$

Note: ① Eq^n of xy plane in $z=0$

$\phi = z$

$\text{grad}\,\phi = i(0) + j(0) + \hat{k}(1) = \vec{k}$



② Directional derivative of $\phi(xyz)$ at a point 'p' in the dir^n of $\vec{n}$ is given by $D.D = (\nabla\phi)_p \cdot \hat{n}$

③

	Initial func	Resultant func
Gradient	Scalar	vector
Divergence	vector	Scalar
curl	vector	vector

④ curl grad $\phi = 0$ for any scalar ϕ

i.e grad ϕ is always an irrotational vector.

⑤ div curl $\vec{F} = 0$ for any vector $\vec{F}$

i.e curl $\vec{F}$ is always a solenoidal vector.

⑥ If ϕ is a scalar & $\vec{F}$ is vector then

$\nabla\cdot(\phi\vec{F}) =$

$$\boxed{\text{div}(\phi A) = \phi \, \text{div} A + \text{grad}\,\phi\cdot\vec{A}}$$

⑦ curl curl $\vec{F}$ = grad div $\vec{F} - \nabla^2\vec{F}$

$$\boxed{\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}}$$

Q If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2+y^2+z^2}$ then find

1) $\nabla\cdot\vec{r}$ 2) ∇r 3) ∇r^n 4) $\nabla^2 r^n$ 5) $\nabla^2(1/r)$

1) $\nabla\cdot\vec{r} = 3$

2) $\nabla r = \hat{r}$

3) $\nabla r^n = n r^{n-1}\hat{r}$

4) $\nabla^2 r^n = \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

$= n(n-1)r^{n-2} + \frac{2}{r} n r^{n-1}$

$= n(n-1)r^{n-2} + 2n r^{n-2} = n r^{n-2}(n-1+2)$

$= n(n+1) r^{n-2}$

5) $\nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{2}{r}\left(\frac{-1}{r^2}\right)$

$\frac{2}{r^3} - \frac{2}{r^3} = 0$

Vector integral

① Line Integral

The line integral of a vector $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ along a curve "C" is given by $\int_C \vec{F}\cdot\vec{dr}$

where $\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

work done : If $\vec{F}$ the force acting on a particle to move along a curve c from a point A to point B then $\int_A^B \vec{F}\cdot\vec{dr}$ denotes the amount of work done to move the particle from A to B.

Q Find the work done when a force $\vec{F} = y\hat{i} + x\hat{j}$ moves along $y = 2x$ from (0 0) to (1 2).

$$\int_A^B \vec{F}\cdot\vec{dr} = \text{work done}$$

$$\int_A^B y\,dx + x\,dy$$

along $y = 2x$

$dy = 2dx$

$$\int_0^1 2x\,dx + x\,2dx = 4\left[\frac{x^2}{2}\right]_0^1 = 2 \text{ units}$$

Q Find work done when a Force $F = x\hat{i} + y\hat{j} + z\hat{k}$ moves a particle along $x = t$, $y = t^2$, $z = t^3$ from (000) to (111)

$$\int \vec{F}\cdot\vec{dr} = \int x\,dx + y\,dy + z\,dz$$

$$\int_0^1 t\,dt + t^2\,2t\,dt + t^3\,3t^2\,dt$$

$$\int_0^1 (t + 2t^3 + 3t^5)\,dt = \frac{t^2}{2} + \frac{2t^4}{4} + \frac{3t^6}{6}\Big|_0^1 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$\frac{3}{2}$ units of work is to be done to move the particle from (0,0,0) to (1,1,1)

Surface Integral :

The surface integral of a vector $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ along a surface S is given by $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\hat{n}$ is the unit outward normal to the surface 'S'

$\hat{n}$ — unit normal outward to the surface

$ds = dx\,dy$ if S is on xy plane

$= dy\,dz$ if " " " yz "

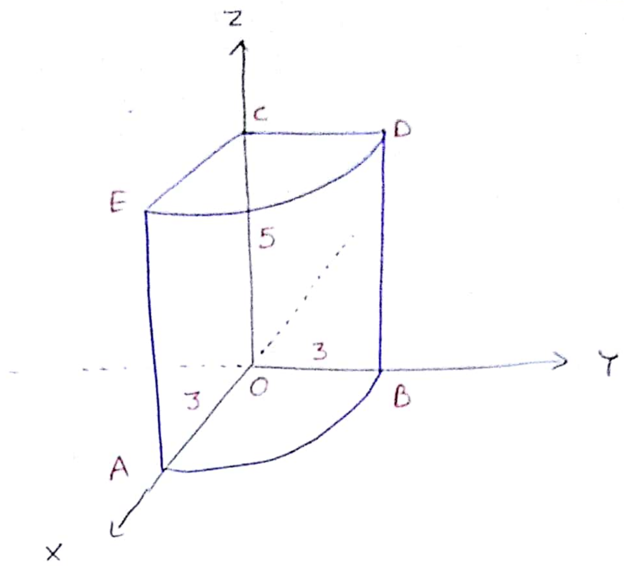
$= dx\,dz$ " " " " xz "

$= \dfrac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$ if S is projected on xy plane

$= \dfrac{dx\,dz}{|\vec{n}\cdot\vec{j}|}$ " " " " " xz plane

$= \dfrac{dy\,dz}{|\vec{n}\cdot\hat{i}|}$ " " " " yz plane

Q The value of $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the closed surface of $\frac{1}{4}$th portion of the cylinder $x^2+y^2=9$, $z=0$, $z=5$ in First octant.



there are 5 different surfaces in the dgm.

Let $S_1 \rightarrow OAB$ is on XY plane

$S_2 \rightarrow OBDC$ is on YZ plane

$S_3 \rightarrow OAEC$ is on xz plane

$S_4 \rightarrow CDE$ is parallel to XY plane

$S_5 \rightarrow ABDE$ is the curved surface of cylinder.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \cdots\cdots + \iint_{S_5} \vec{F}\cdot\hat{n}\,ds$$

Along S_1 : $z=0$ $\hat{n} = -\hat{k}$

$\vec{F}\cdot\hat{n} = -z = 0$; $ds = dx\,dy$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} 0\,dx\,dy$$

$$= 0$$

Along S_2 : $\hat{n} = -\hat{i}$ $x=0$ $ds = dy\,dz$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint 0.\,dx\,dy = 0$$

along S_3 : $y=0$ $\hat{n} = -\hat{j}$ $ds = dx\,dz$

$\vec{F}\cdot\hat{n} = -y = 0$ $(x\hat{i} + y\hat{j} + z\hat{k})$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = 0$$

Along S_4 : CDE $z=5$ $\hat{n} = \hat{k}$ $ds = dx\,dy$

$\vec{F}\cdot\hat{n} = z = 5$

Here we willl use projection.

projected plane.

$$\iint 5\,.dx\,dy =$$

$$ds = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

now S_1 will become S_4

S_1 because projecting S_4 in xy plane i.e S_1

$s = dx\,dy$

$$\iint_{S_1} 5\,dx\,dy = 5\left(\frac{\pi r^2}{4}\right)$$

$$= \frac{5\pi\,3^2}{4}$$

$$= \frac{45\pi}{4}$$

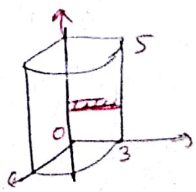
Along S_5 : ABDE surface

take any point on surface it will satisfy $x^2+y^2=9$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{3}$$

$\vec{F}\cdot\hat{n} = \frac{1}{3}[x^2+y^2] = \frac{9}{3} = 3$ if S_5 is projected on yz plane

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dy\,dz}{(x/3)} = \frac{3\,dy\,dz}{\sqrt{9-y^2}}$$



$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} 3\cdot\frac{3}{\sqrt{9-y^2}}\,dy\,dz = \int_{z=0}^{5}\int_{y=0}^{3} \frac{9}{\sqrt{9-y^2}}\,dy\,dz$$

$$= \int_{z=0}^{5} \left(9 \cdot \sin(y/3)\right)_0^3 \, dz = \frac{9\pi}{2} \times (5) = \cancel{\frac{45}{2}\pi}\ \frac{45\pi}{2}$$

$$0 + 0 + 0 + \frac{45\pi}{4} + \frac{45\pi}{2} = \frac{135\pi}{4}$$

By Gauss divergence theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \operatorname{div} \vec{F}\, dv = \iiint_V 3\, dv = 3\left[\frac{1}{4}\pi 3^2 \cdot 5\right] = \frac{135\pi}{4}$$

Volume Integral

The volume integral of a vector $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ along a volume V is given by

$$\iiint_V \vec{F}\, dv = \hat{i}\iiint_V F_1\, dv + \hat{j}\iiint_V F_2\, dv + \hat{k}\iiint_V F_3\, dv$$

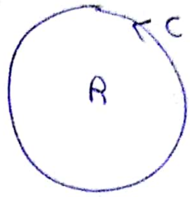
Green's Theorem : Relation b/w line integral and surface integral

If $M(x,y)$ & $N(x,y)$ are two continuous func in a region R bounded by a closed curve C in xy plane then (yz plane, zx)

limitⁿ – the surface always may not be xy, yz, and zx plane so come with Stoke theorem.

$$\oint M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

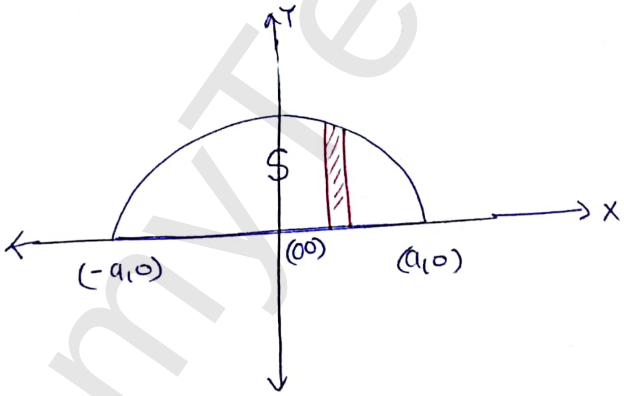
$$\oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$



Q the value of $\oint_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$ where C is the upper half of the circle $x^2+y^2 = a^2$ bounded by x axis.

By Greens theorem $\oint M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_S (2x + 2y)\,dx\,dy$$



$$= \int_{-a}^{+a} \int_{y=0}^{\sqrt{a^2-x^2}} (2x+2y)\,dy\,dx$$

$$= \int_{-a}^{+a} \left[x(y)_0^{\sqrt{a^2-x^2}} + \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} \right] dx$$

$$= 2\int_{-a}^{a} \left[\cancel{x\sqrt{a^2-x^2}}^{\,0} + \frac{(a^2-x^2)}{2} \right] dx$$

$\downarrow$ odd.

$$= a^2(x)_{-a}^{a} - \left(\frac{x^3}{3}\right)_{-a}^{a} = 2a^3 - \frac{2a^3}{3} = \frac{4a^3}{3}$$

Stokes Theorem: Relation b/w line integral & Surface integral

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is any vector point function in an open region S, bounded by a closed curve C then

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

Note* By considering $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ & S is a surface on xy plane

$$\text{LHS} = \oint_C \vec{F} \cdot \vec{dr} = \oint M\,dx + N\,dy$$

For RHS :: $\hat{n} = \vec{k}$ $\quad \text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$

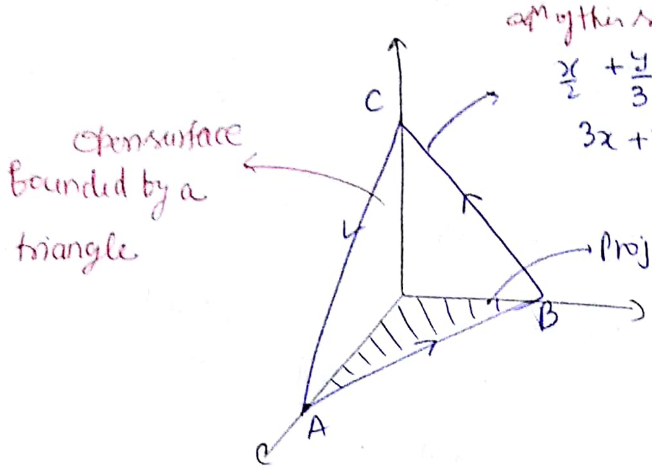
$$\text{curl}\,\vec{F} \cdot \hat{n} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \quad - \quad ds = dx\,dy$$

$$\text{RHS} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Q The value of $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

where C is the Δ^{le} vertices (2 0 0) (0 3 0) (0 0 6)

Sol^n



$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$3x + 2y + z = 6$

either i do line integral frm AB then BC and then CA thru times. or i use stokes theorem: Stoke theorem used for open surfaces. now i calculated $\hat{n}$, calc^c curl $\vec{F}$ now i need to find projected plane ds = dxdy

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\hat{i}+2\hat{j}+k}{\sqrt{9+4+1}} = \frac{3\hat{i}+2\hat{j}+k}{\sqrt{14}}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = i(1+1) - j(0-0) + k[2-1]$$

$$= 2i + k$$

$$\text{Curl}\,\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}}$$

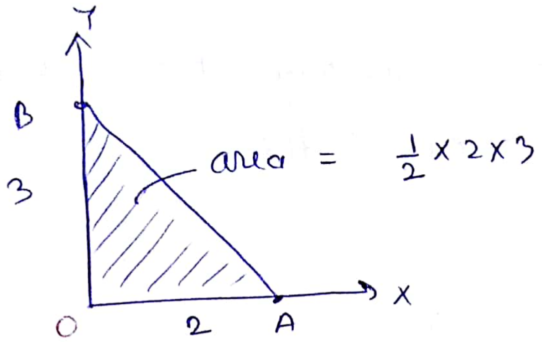
if S is projected on xy plane $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{(1/\sqrt{14})}$

By stokes $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds = \iint_R \frac{7}{\sqrt{14}} \times \frac{dx\,dy}{(1/\sqrt{14})} = 7\iint_R dx\,dy$

work done

$= 7\left[\frac{1}{2}\times 2\times 3\right] = 21$ unit

Closed curve C is not here on one plane so work done is not zero i.e 21.



when we move on a closed curve work done = 0 ~~bez~~but that curve should be in a single plane if we move A O A to O to B to A then work done $\oint \vec{F} \cdot dr = 0$ but Here we are moving in a curve which is not in a plane that's why $\oint \vec{F} \cdot dr = 21$

Stokes is a modification of green theorem.

Gauss Diver — Relation b/s Sur Integral and Volume Integral

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is any vector point func in a closed region S enclosing a volume V then $\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\mathrm{div}\,\vec{F})\,dv$

Q The value of $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = x\hat{i} + y\hat{j} + 3\hat{k}$ and S is closed surface of

(1) the Rectangular box $x = \pm 1$ $y = \pm 2$ $z = \pm 3$

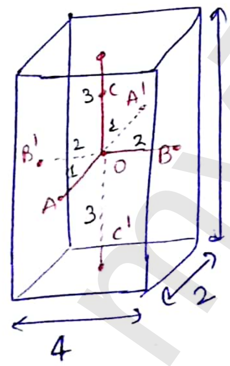
(2) closed surface of upper half of sphere $x^2+y^2+z^2 = 25$ bounded by xy plane

(3) of a volume V.

By G.D. Th

$$\oiint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\vec{F}\,dv = 3\iiint_V dv$$

(1)



$V = 4\times2\times6$

$V = 48$

$3\iiint_V 48 = \quad 3\times48$

(2) $3\times\left[\frac{1}{2}\left(\frac{4}{3}\pi 5^3\right)\right]$

(3) $3\times\iiint dv$

$3\times V = 3V$

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* if $\vec{F} = x\hat{i} + y\hat{j}$

take care

then $= 2\iiint_V dv$

in place of $\vec{F}$

any vector point func

* if $\iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$ where $\vec{F}$ and S is any closed surface is

$$\oiint_S \text{curl}\vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}(\text{curl}\vec{F})\,dv = 0$$

* Green and stoke relate line Integral to Surface Integral

* No therem to relate b/w line Integral and volume Integral.

$$\oint_C \vec{F}\,dr = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds \neq \iiint_V \text{div}(\text{curl}\vec{F})\,dv$$

Complex Variable

A variable of the form $z = x + iy$ $\quad i^2 = -1$, x, y are real

By taking $x = r\cos\theta$, $y = r\sin\theta$

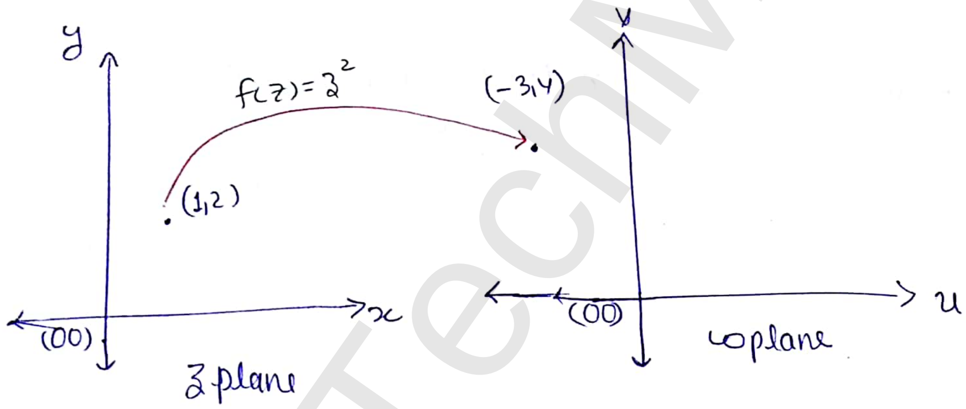
$$z = re^{i\theta}$$

$$z = x + iy$$

$$z = r\cos\theta + ir\sin\theta$$

$$|z| = r = \sqrt{x^2+y^2} \qquad \theta = \text{amp } z = \tan^{-1}\frac{y}{x}$$

$$w = f(z) = u(x,y) + i\,v(x,y)$$



Some basic func.

(1) $\quad f(z) = z^2$

$$= (x+iy)^2 =$$

$$= (x^2 - y^2) + i\,2xy$$

$x = 1,\ y = 2$

$(1,2) \longrightarrow (-3,4)$

$(x,y) \qquad (u,v)$

(2) $f(z) = e^{z}$

$= e^{x+iy}$

$= e^{x} \cdot e^{iy}$

$= e^{x}[\cos y + i \sin y]$

$= e^{x}\cos y + i e^{x}\sin y$

(3) $f(z) = \frac{1}{z} = \frac{1}{x+iy}$

$\frac{x-iy}{x^2+y^2}$

$= \left[\frac{x}{x^2+y^2}\right] + i\left[\frac{-y}{x^2+y^2}\right]$

(4) $f(z) = \ln z$

$= \ln re^{i\theta}$

$= \ln r + \ln e^{i\theta}$

$= \ln\sqrt{x^2+y^2} + i\theta \ln e$

$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

5) $f(z) = z^3$

$= (x+iy)^3$

$= x^3 - iy^3 + 3x^2 iy - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

(6) $f(z) = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$\theta = iy$

$\cos iy = \frac{e^{-y}+e^{y}}{2}$

$= \cosh y$

$\sin iy = \frac{e^{-y}-e^{y}}{2i} \times \frac{i}{i}$

$= i\,\frac{e^{y}-e^{-y}}{2}$

$= i\sinh y$

$\boxed{\sin z = (\sin x \cosh y) + i(\cos x \sinh y)}$

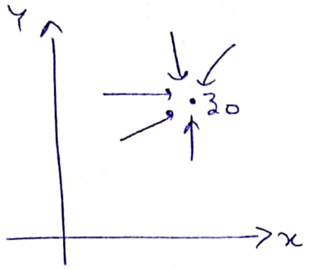
7) $\cos z = \cos(x+iy)$

$= \cos x \cos iy + \sin x \sin iy$

$$\boxed{\cos z = \cos x \cosh y + i[-\sin x \sinh y]}$$

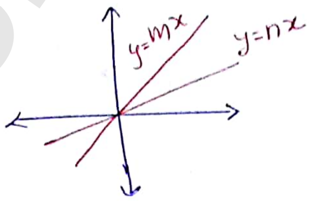
Limit of $f(z)$:-

A complex fun^c $f(z)$ is said to have limit l as z tends to z_0 in any direction if $\lim_{z \to z_0} f(z) = l$



Z plane

Ex. $\lim_{z \to 0} \frac{xy}{x^2+y^2}$ doesn't exist

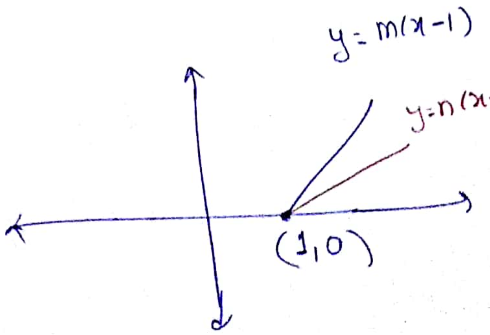


$z \to 0$ along $y = mx$

$$\lim_{z \to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

$z \to 0$ along $y = nx$

$$\lim_{z \to 0} \frac{nx^2}{x^2+n^2x^2} = \frac{n}{1+n^2}$$



2) $\lim_{z \to 1} \frac{xy}{x^2+y^2}$ exist

$z \to 1$ along $y - 0 = m(x-1)$

$$\lim_{z \to 1} \frac{x \cdot m(x-1)}{x^2 + m(x-1)^2} = \frac{0}{1} = 0$$

if slope of line $= n$ then also $\lim_{z \to 1} f(z) = 0$

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it is coming 0 every time bcz of $(x-1)$ factor.

Continuity of $f(z)$: A complex funᶜ $f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$

continious होने के लिए limit exist करना जरुरी है. पर limit exist कर रही है इसका मतलब जरुरी नहीं कि continious हो

Differentiability of $f(z)$

in Real ~~plane~~ function.

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f(x) = x^2 \qquad f'(x) = 2x \qquad f''(x) = 2$$

$$= \lim_{x \to a} \frac{x^2 - a^2}{x - a} = (x+a)$$

$$= 2a$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} = \lim_{\delta x \to 0} \frac{\cancel{x^2} + \delta x^2 + 2x\delta x - \cancel{x^2}}{\delta x} = 2x + \cancel{\delta x}^{\,0}$$

$$= 2x$$

a complex funᶜ $f(z)$ is said to be differentiable at $z = z_0$ if ~~the limit~~ value ~~$z \to z_0$~~

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ exist and is denoted by $f'(z_0)$

Analytic function (this def^n exist for a comp fun^c only)

A comp fun^c $f(z)$ is said to be analytic at a point $z = z_0$ if the derivative of $f(z)$ exists not only at z_0 but also in some neighbourhood of z_0

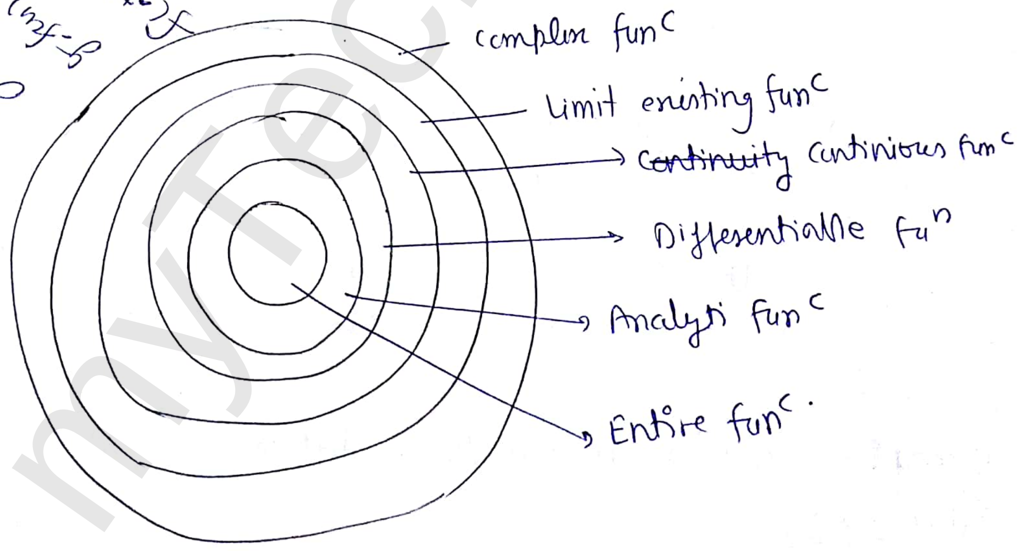
A complex fun^c $f(z)$ is said to be analytic in a region R of z plane if the derivative of $f(z)$ exist throughout the region R.

if a fun^c is analytic — means differentiable

if fn diff'le ——— need not analytic

Entire fun^c

A complex fun^c $f(z)$ is said to be an entire fun^c or a holomorphic or a regular fun^c if the derivative of $f(z)$ exist throughout the z plane..



$x = a^+$

Continuity

$L.H.L = R.H.L = f(a)$

$f(a^-) = f(a^+) \neq f(a)$

Similarly we say for Real fun^c.

$f(z_0)$ — Analytic if derivative exist in neighbourhood of z_0.

Necessary and Sufficient Condⁿ for $f(z) = u(x,y) + i v(x,y)$ to be analytic in a regular R of z plane are

(1) The partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ must exist in the Region R

2) they should satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ } Cauchy Reiman eqⁿ & or CR eqⁿ

ex:- $f(z) = z^2$ Find f(z) is analytic or not

$= (x^2 - y^2) + i\, 2xy$

$u = x^2 - y^2 \qquad V = 2xy$

$u_x = 2x \qquad V_x = 2y$

$u_y = -2y \qquad V_y = 2x$

$\boxed{u_x = V_y} \quad \boxed{u_y = -V_x}$

$f(z) = z^2$ is analytic in all region of z plane so it is entire func.

ex: $f(z) = \bar{z}$

$= x - iy$

$u = x \qquad V = -y$

$u_x = 1 \qquad V_x = 0$

$u_y = 0 \qquad V_y = -1$

$u_x \neq V_y$ (for any z) $u_x \neq V_y$ so it is not analytic func. not analytic through out the z plane

③ $f(z) = z^2 + \bar{z}$

$= (x^2 - y^2 + x) + i\,(2xy - y)$

$u = x^2 - y^2 + x$ $\quad v = 2xy - y$

$u_x = 2x + 1$ $\quad v_x = 2y$

$u_y = -2y$ $\quad v_y = 2x - 1$

$u_x \neq v_y$

$f(z) = z^2 + \bar{z}$ is not analytic.

Note *

(1) Any fun^c which involve $\bar{z}$ is always not analytic

(2) Every polynomial fun^c in z is an entire function.

④ $f(z) = e^z$

$= e^x \cos y + i(e^x \sin y)$

$u = e^x \cos y$ $\quad v = e^x \sin y$

$u_x = e^x \cos y$ $\quad v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\quad v_y = e^x \cos y$

$u_x = v_y$, $u_y = -v_x$

$f(z) = e^z$ is also an entire fun^c

every exponential fun^c is entire fun^c

Ex. $f(z) = u + iv$ ~~for which~~ Find fun^c analytic or not

$u_x = \frac{1}{x-2}$ $v_x = -1$

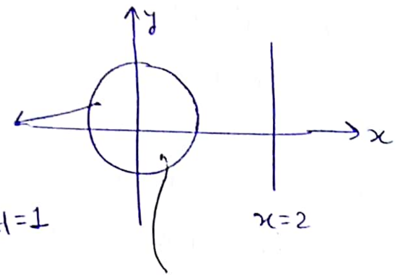
$u_y = 1$ $v_y = \frac{1}{x-2}$

Sol^n

For a fun^c tobe Analytic 1st cond^n is that the partial derivative must exist in the complete complex Region. but Here for some regions where $x=2$, u_x & v_y don't exist so this fun^c is not analytic in some part of Z plane.

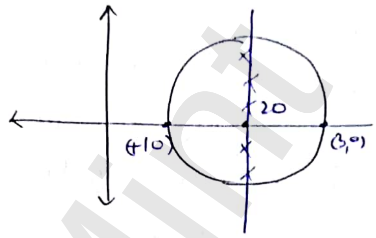
Cauchy R eq (1) not satisfied anytime

Let take Region R_1



so for this Region R_1 fun^c analytic

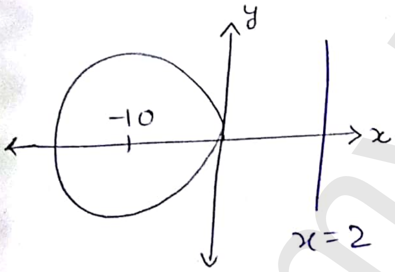
Region R_3 is $|z-2| = 1$



for Region R_3 fun^c is not analytic bcz u_x and v_y don't exist

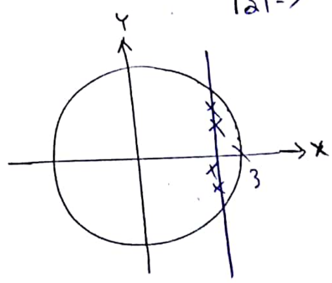
R_2 Region

$|z+1| = 1$



R_4 Region $|z|=3$

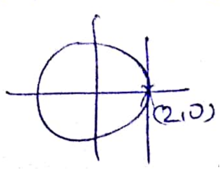
$|z|=3$



this fun^c is analytic in the Region which doesn't contain the line $x=2$

this fun^c is not analytic in the Region which contains the line $x=2$ such as $|z| = 1$, $|z+1| = 1$

Here we don't discuss about when point is on the circle, but actually that is also not analytic.

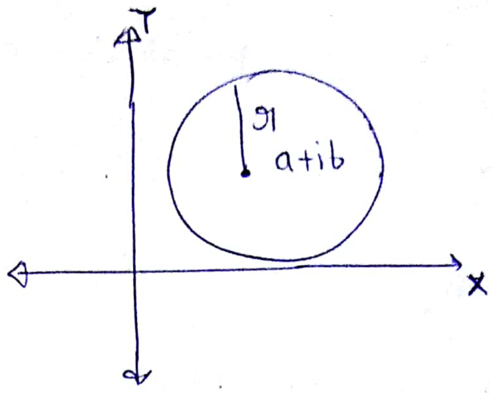


$|z - (a+ib)| = r$

$|(x+iy) - (a+ib)| = r$

$|(x-a) + i(y-b)| = r$

$(x-a)^2 + (y-b)^2 = r$



Note* → Regular functions are either analytic or non-analytic

→ The type of func's which are analytic at some points are not generally used or are not regularly used.

→ So Generally whenever we see a func is analytic means, we think it is analytic in the complete complex Region i.e. its an entire function

C

→ So generally we can assume that analytic functions are entire func's.

CR eq^{n}s in polar form

we have $f(z) = u + iv$

$$f(re^{i\theta}) = u(r\theta) + i\,v(r\theta) \quad \text{—— (1)}$$

differentiate (1) partially wrt ~~x~~ r

$$f'(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

differentiate (1) partially wrt θ

$$f'(re^{i\theta})\, r\, i\, e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$-r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} \quad \text{and} \quad r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

$$\boxed{u_r = \frac{1}{r} v_\theta} \qquad \boxed{v_r = -\frac{1}{r} u_\theta}$$

CR eq^{n}s in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Properties :- If $f(z) = u + iv$ is analytic func then

① $u(x,y) = c_1$ & $v(x,y) = c_2$ are orthogonal to each other.

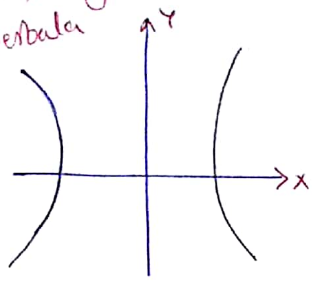
Ex: $f(z) = z^2$

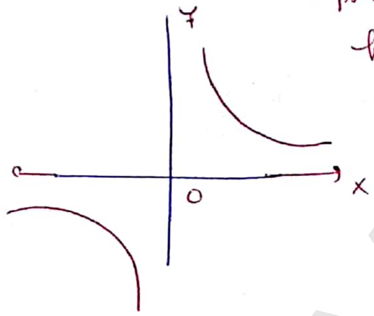
$= (x^2 - y^2) + i\,2xy$

$x^2 - y^2 = c_1$ $\quad$ $2xy = c_2$

represent a family of hyperbola

represent a family of Rectangular hyperbola.





means these two curves cut each other ~~at~~ perpendicularly.

② $u(x,y)$ is a harmonic func & $v(x,y)$ is a harmonic func

Harmonic func: A func $H(x,y)$ satisfies laplace eqn $\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$ is called a harmonic func.

Since $f(z) = u + iv$ is analytic func

$u_x = v_y$ $\quad$ $u_y = -v_x$ $\quad$ again partial diff wrt x

$u_{xx} = v_{xy}$ —— ①

partial differentiating wrt y.

$u_{yy} = -v_{yx}$ —— ②

add ① + ②

$u_{xx} + u_{yy} = v_{xy} - v_{yx}$ $\qquad$ $\boxed{v_{xy} = v_{yx}}$

$\boxed{u_{xx} + u_{yy} = 0}$

Similarly

$\boxed{v_{xx} + v_{yy} = 0}$

Q If $f(z) = u+iv$ is a analytic func for which $u = y + e^x \cos y$ then

$v =$ ______

Soln

$u_x = e^x \cos y$

$u_y = 1 \mp e^x \sin y$

a) $x + e^x \sin y + \text{const} \xrightarrow{v_x} 1 + e^x \sin y$

b) $-x + e^x \sin y + \text{const} \xrightarrow{v_x} -1 + e^x \sin y$

$\xrightarrow{v_y}\ 0 + e^x \cos y$

c) $x - e^x \sin y + \text{const}$

d) $-x - e^x \sin y + \text{const}$

Integration method to find v

$f(z) = u + iv$

$f'(z) = u_x + i v_x$

$f'(z) = u_x - i u_y$

By Sir.

$u_x = e^x \cos y = v_y$

$u_y = 1 - e^x \sin y$

$\frac{\partial v}{\partial y} = e^x \cos y$

integrate partialy w.r.t y treating x as const

$v = e^x \sin y + g(x)$

$\frac{\partial v}{\partial x} = e^x \sin y + g'(x) = -\frac{\partial u}{\partial y} = -1 + e^x \sin y$

$\therefore g'(x) = -1 \Rightarrow g(x) = -x + \text{const}$

$v = e^x \sin y - x + \text{const}$

Q If $f(z) = u+iv$ is analytic func for which $u = 2xy$ then $v =$ ____

(a) $x^2 - y^2 + const$ $\longrightarrow V_x = 2x$

b) $y^2 - x^2 + const$ $\longrightarrow V_x = -2x$, $V_y = 2y$

c) $x^2 + y^2 + const$

d) $-x^2 - y^2 + const$

Soln $\frac{\partial u}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = 2x$

option (b)

Q $f(z) = u+iv$ is analytic func then $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 =$ ____

a) $|f'(z)|^2$

b) $2|f'(z)|^2$

c) $2|f(z)|^2$

d) $2|f'(z))^2$

soln $|Re\ f(z)|^2 = u^2$

$\frac{\partial u^2}{\partial x} = 2u \frac{\partial u}{\partial x}$

$\frac{\partial^2 u^2}{\partial x^2} = \frac{\partial}{\partial u}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$

$\frac{\partial^2 (u^2)}{\partial y^2} = \qquad = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right]$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \cancel{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}\right] = 2|f'(z)|^2$

so Harmonic func

ok, How this equal to $|f'(z)|^2$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial u}{\partial y}\right)^2$$

option (b)

Q If $z = x+iy$ then $|e^{iz}| =$

a) $=1$

b) e^{y}

c) $e^{\sqrt{x^2+y^2}}$

d) e^{-y}

$$|e^{i(x+iy)}|$$

$$|e^{ix} \cdot e^{-y}|$$

$$|e^{ix}|\,|e^{-y}|$$

$$= |\cos x + i\sin x|\,|e^{-y}|$$

$$= 1 \cdot e^{-y} \longrightarrow \text{is always a +ve quantity}$$

option (d)

why not option (c)

Q $f(z) = z^2$ maps the first quadrant into which part of ω plane

a) 1st quadrant

b) upper half plane

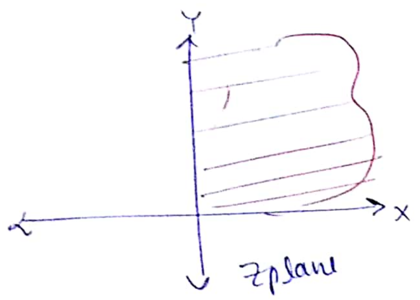
c) Lower half plane

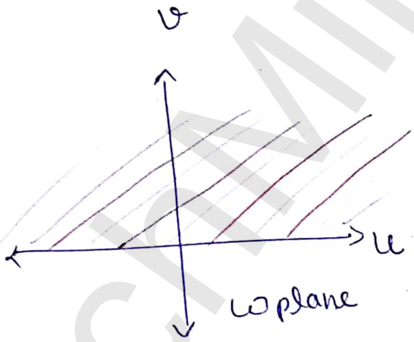
d) to $z \to \infty$

$f(z) = (x+iy)^2$

$f(z) = (x^2-y^2) + i\,2xy$

$u = x^2-y^2 \qquad v = 2xy$





b/c "v" will always be +ve.

Q Let $S = \{z : |z| = 1\}$

$f(z) = z\bar{z}$ then $f(z)$ maps S to which of the following in ω plane

S: a) unit circle

b) horizontal axis line segment

(c) From origin to (1,0)

c) The point (1,0)

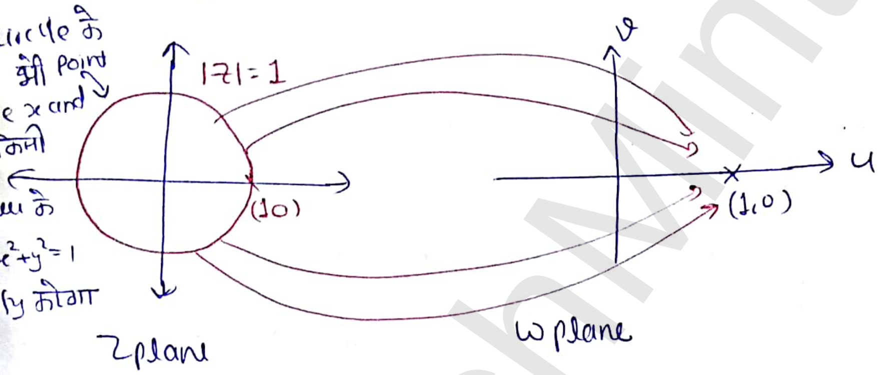
d) The entire horizontal axis.

$$f(z) = (x+iy)(x-iy)$$

$$= (x^2+y^2) + i(0)$$

$u = x^2+y^2 \qquad v = 0$

इस circle के किसी भी point पे (i.e x and y की किसी भी value के लिए) $x^2+y^2=1$ satisfy होगा



if i want my option to be (b) then $f(z) = z\bar{z} < 1$

then our answer will be option (b).

Complex Integration

An integral of the form

$$\int_C f(z)\,dz \quad \text{where } dz = dx + i\,dy \text{ is called a complex Integral}$$

we will discuss
4 model

(Q) The value of $\int_0^{2+i} \bar{z}\,dz$ along 1) $y = x/2$
2) The real axis to 2 and then vertically to $2+i$

1) Along $y = \frac{x}{2}$
$dy = \frac{dx}{2}$

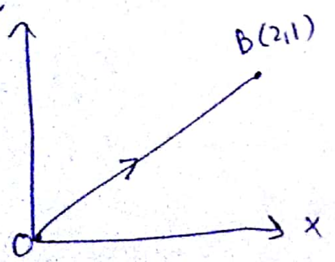
model 1 open curve and Non-analytic func -
then change Total func into single variable

$$\int_0^{2+i} (x - iy)(dx + i\,dy) = \int_0^2 \left(x - \frac{ix}{2}\right)\left(dx + i\frac{dx}{2}\right) = (1 - i/2)(1 + i/2)\left(\frac{x^2}{2}\right)_0^2$$

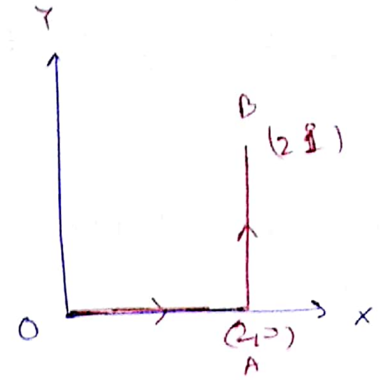
$$= \left(1 + \frac{1}{4}\right)(2) = \frac{5}{2}$$

Total func in y variable.

$$\int_0^1 (2y - iy)(2dy + i\,dy) = (2-i)(2+i)\left(\frac{y^2}{2}\right)_0^1 = 5 \times \frac{1}{2} = \frac{5}{2}$$



(2)



$$\int_0^{2+i} \bar{z}\, dz = \int_{\text{along } OA} \bar{z}\, dz + \int_{\text{along } AB} \bar{z}\, dz$$

Along OA :

OA is x axis so $y=0$; x varies from 0 to 2

$$\int_{\text{along } OA} (x-iy)(dx+idy) = \int_0^2 (x-0)(dx+0) = \left.\frac{x^2}{2}\right|_0^2 = 2$$

Along AB: $x=2$; $dx=0$ y varies from 0 to 1

$$\int_{\text{along } AB} (x-iy)(dx+idy) = \int_0^1 (2-iy)(0+idy) = i\left[2(y)_0^1 - i\left(\frac{y^2}{2}\right)_0^1\right] =$$

$$= i\left[2-\frac{i}{2}\right] = \frac{1}{2}+2i$$

when $f(z)$ is not analytic func the $\int$ O to A to B is $\frac{1}{2}+2i+2$ and $\int$ O to B directly as in (1) is $\frac{5}{2}$ means: if path changes the value of Integration also changes.

For non analytic $\int$ depends on path.

Q The value of $\int_{1-i}^{2+i} x\, dz$ along $x = t+1$, $y = 2t^2 - 1$

Solⁿ $f(z) = x$ Non analytic and open curve so change in op. single variable.

$dx = dt \qquad dy = 4t\,dt$

x varies from 1 to 2

y " " -1 to 1 not from $-i$ to i

U.L when $x = 2 \quad t = 1$ ~~U.L when~~ $y =$ when $y = 1$ $\quad t = \pm 1$

L.L " $x = 1 \quad t = 0$ when $y = -1 \quad t = 0$

ambiguity, which one to take +1 or −1. So don't define limit here using y.

$$\int_{1-i}^{2+i} x(dx + i\,dy)$$

$$\int_0^1 (t+1)\left[dt + i\,4t\,dt\right]$$

$$\therefore \int_0^1 \left[(t+1) + i(4t^2 + 4t)\right] dt$$

$$= \left(\frac{t^2}{2}\right)_0^1 + (t)_0^1 + i\left[4\left(\frac{t^3}{3}\right)_0^1 + 4\left(\frac{t^2}{2}\right)_0^1\right]$$

$$= \frac{1}{2} + 1 + i\left[\frac{4}{3} + 2\right]$$

$$= \frac{3}{2} + \frac{10}{3} i$$

M2 Analytic func + open curve. then solve like a usual integration

Q The value of $\int_{1-i}^{2+i} z\, dz$ along $x = t+1$ $y = 2t^2-1$

{ Regular func's are all entire func }

$$= \left(\frac{z^2}{2}\right)_{1-i}^{2+i}$$

$$= \frac{1}{2}\left[(2+i)^2 - (1-i)^2\right]$$

{ independent of the path of the integration }

m3: Closed curve + Not analytic function

Note* (1) when the curve c is an open curve and $f(z)$ is a non analytic func then we have to convert the total func in terms of one single variable.

(2) when the curve c is a open curve and $f(z)$ is an an entire func then we can evaluate it as a usual func of z

Model-3

Q. The value of $\oint_C Re(z)\, dz$ where c is the unit circle $|z| = 1$

{ closed curve + non analytic func } so solve using polar coordinate

$x = r\cos\theta$ $y = r\sin\theta$

$z = re^{i\theta} = e^{i\theta}$ ($\because r = 1$ here)

$dz = ie^{i\theta}\, d\theta$ $x = \cos\theta$

0 to 2π bcz unit circle

$$\oint_C Re\, z\, dz = \int_0^{2\pi} \cos\theta\, (ie^{i\theta})\, d\theta$$

$$= \int_0^{2\pi} i(\cos\theta)\cdot(\cos\theta + i\sin\theta)\, d\theta$$

$$= i\left[\int_0^{2\pi} \cos^2\theta\, d\theta + i\int_0^{2\pi} \cos\theta\sin\theta\, d\theta\right]$$

$$\left\{ \begin{array}{l} \int_0^{2\pi} \sin mx\, dx = 0 \\ \int_0^{2\pi} \cos mx\, dx = 0 \end{array} \right.$$

$$= i\left[\frac{1}{2}(\theta)_0^{2\pi}\right] = \pi i$$

$2\cos^2\theta - 1 = \cos 2\theta$

Model 4 Curve closed + Analytic fun^c

Cauchy's Integral theorem:

If $f(z)$ is analytic inside and on a closed curve c then $\oint_c f(z)\, dz = 0$

Ex. $\oint_{|z|=1} (z^2 + 3z + 5)\, dz = \cancel{0} = ?$

$\underbrace{(z^2+3z+5)}_{\text{analytic bcz polynomial of } z}$

$= 0$

Ex. $\oint_{|z|=1} \dfrac{z^2+3z+5}{z-2}\, dz$

= will be 0

Ex. $\oint_{|z|=3} \dfrac{z^2+3z+5}{z-2} =$

"Cauchy Integral theorem not applicable. bcz limit don't exist at $z=2$"

Cauchy's Integral formula: If $f(z)$ is analytic inside and on a closed curve C and $z = a$ is a point inside 'C' then

$$\oint \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i \frac{f^{n}(a)}{\lfloor n}$$ differentiation

Ex:- $$\oint_{|z|=3} \frac{z^2+3z+5}{z-2} dz$$

$$= 2\pi i \frac{f^{(0)}(2)}{\lfloor 0}$$ where $f(z) = z^2+3z+5$

$f(2) = 4+6+5$
$= 15$

$$= 2\pi i \times 15$$

$$= 30\pi i$$ Answer.

(2) $$\oint_{|z|=3} \frac{z^2+3z+5}{(z-2)^2} dz = \frac{2\pi i}{\lfloor 1} \times 2z+3 \Big|_{z=2}$$

$$= 2\pi i \cdot 7 = 14\pi i$$

(3) $$\oint_{|z|=2} \frac{e^{2z}}{(z+1)^3} dz$$ ← while solving these questions remember these all are model 4 i.e $\boxed{C+A}$ $\boxed{\text{Closed + Analytic}}$
$f(z)$ = Analytic.

$$4 \quad z \to -1 \quad \frac{2\pi i}{\lfloor 2} \frac{d^2}{dz^2} e^{2z} \Big|_{z=-1}$$

$$= \frac{2\pi i}{2} e^{2z} \cdot 2 \cdot 2 \Big|_{z=-1}$$

$$= 4\pi i e^{-2}$$

$$= \frac{4\pi i}{e^2}$$ Ans

Q $\oint_C \frac{2z+1}{z^2+z} dz$ where C is $|z| = \frac{1}{2}$

$$\oint_C \frac{\frac{2z+1}{z+1}}{z} dz = 2\pi i\, f(0) \quad \text{where } f(z) = \frac{2z+1}{z+1}$$

$$= 2\pi i \qquad f(0) = 1$$

Q $\oint_C \frac{z}{z^2-3z+2} dz$ C is $|z-2| = \frac{1}{2}$

C is $|z-2| = \frac{1}{2} \Rightarrow (x-2)^2 + y^2 = \frac{1}{4}$

$S \equiv (x-2)^2 + y^2 - \frac{1}{4}$

when $z = 1 \Rightarrow x=1, y=0$

$S \equiv 1 + 0 - \frac{1}{4} = \frac{3}{4} > 0$ outside

when $z = 2 \Rightarrow x = 2, y = 0$

$S \equiv 0 + 0 - \frac{1}{4} < 0$ inside

$$\oint_C \frac{\left(\frac{z}{z-1}\right)}{z-2} dz = 2\pi i\, f(2) \qquad f(z) = \frac{z}{z-1}$$

$$= 4\pi i \qquad f(2) = 2$$

Q. $\oint_c \frac{z-3}{z^2+2z+5}\,dz$ where c is $|z+1-i| = 2$

$z = \frac{-2\pm\sqrt{4-20}}{2} = -1\pm 2i$

=

$|z+1-i| = 2 \Rightarrow (x+1)^2+(y-1)^2 = 4$

$S \equiv (x+1)^2+(y-1)^2-4$

when $z = -1+2i \Rightarrow x=-1,\ y=2$

$S \equiv 0+1-4 = -3 < 0$ inside

when $z = -1-2i \Rightarrow x=-1,\ y=-2$

$S \equiv 0+9-4 = 5 > 0$ outside

$\oint_c \frac{\frac{z-3}{(z-(-1-2i))}}{z-(-1+2i)}\,dz = 2\pi i\, f(-1+2i)$ when $f(z) = \frac{z-3}{z-(-1-2i)}$

$= 2\pi i\left(\frac{2i-4}{4i}\right)$

$f(-1+2i) = \frac{-1+2i-3}{-1+2i+1+2i}$

$= \pi(i-2)$

$= \frac{2i-4}{4i}$

Q the value of $\oint_c \frac{4z^2+z+5}{z-a}\,dz$ where c is the ellipse $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2 = 1$

if (1) $a = i$

(2) $a = 3.5$

$S \equiv \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2-1$

① if $a=i$. $\oint_c \frac{4z^2+z+5}{z-i}\,dz$

when $z=i \Rightarrow x=0,\ y=1$

$S \equiv 0+\frac{1}{9}-1 = -\frac{8}{9} < 0$ inside

$= 2\pi i\, f(i)$

where $f(z) = 4z^2+z+5$

$f(i) = -4+i+5 = 1+i$

$= 2\pi i\,(1+i)$

$= 2\pi(i-1)$

(ii) If $a = 3.5$

$$\oint_C \frac{4z^2+z+5}{z-3.5}\,dz = 0 \quad \text{by C.I.T}$$

Q The value of $\oint_C \dfrac{z^2-z+4i}{z+2i}\,dz$ where C is $|z|=3$ ______

$= 2\pi i\, f(-2i)$ where $f(z) = z^2 - z + 4i$

$= (-2i)^2 + 2i + 4i$

$= -4 + 6i$

$= -4+6i$

$= 6i-4$

$= 2\pi i\,(6i-4)$

$= 4\pi i(3i-2)$

$= 4\pi(-3-2i)$

$= -4\pi(3+2i)$

Residue theorem:

If $f(z)$ is analytic inside and on a closed curve 'C' except at a finite no. of singularities inside C then $\oint_C f(z)dz = 2\pi i$ (sum of residues over the singularities)

Singularity: A point where the fun^n $f(z)$ fails to be analytic is called called singularity or singular point

Ex $f(z) = \frac{1}{(z-1)(z+2)}$

$z=1$ and $z=-2$ are called singular point i.e singularities of $f(z)$

if $|z| = 1.5$ then $z=1$ is the singularity only. not $z=-2$

Singularity are of 4 type

1. Simple pole : $f(z) = \frac{1}{(z-1)(z+2)}$ here $z=1$ and $z=-2$ are simple poles of $f(z)$

2. pole of order n : $f(z) = \frac{1}{(z-1)^2 (z+2)^3}$ here $z=1$ is a pole of order 2, $z=-2$ is a pole of order 3

3. Essential singularity $f(z) = \sin\left(\frac{1}{z-2}\right) = \frac{1}{z-2} - \frac{1}{(z-2)^3 3!} + \ldots$

max^m power of $\frac{1}{(z-2)^n}$ is $n \to \infty$ can't be defined that's why called $z=2$ is called Essential singularity.

4. Removable singularity

$f(z) = \frac{z^2-1}{z^2-3z+2} = \frac{z+1}{z-2}$ here $z=1$ is a removable singularity and $z=2$ is a simple pole.

singularity which looks like a singularity but can be removed by adjusting the fun^n

Calculation of Residue :

(1) If $z = a$ is a simple pole of $f(z)$ then $\left[\text{Res } f(z)\right]_{z=a} = \lim_{z\to a}\{(z-a)f(z)\}$

(2) If $z = a$ is a pole of order n of $f(z)$ then

$$\left[\text{Res } f(z)\right]_{z=a} = \lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n f(z)\right\}$$

Q the value of $\oint_c \frac{z^2+1}{z^2-2z}\, dz$ where c is $|z| = 3$ is ____

soln

Here $z=0$ and $z=2$ both lie inside $|z|=3$ and are simple poles

$\therefore \oint_c$

$$\text{Res } f(z)\Big|_{at\ z=0} = \lim_{z\to 0}\left\{\cancel{(z-0)}\frac{z^2+1}{\cancel{(z-0)}(z-2)}\right\} = -\frac{1}{2}$$

$$\text{Res } f(z)\Big|_{at\ z=2} = \lim_{z\to 2}\left\{\frac{(z-2)\times(z^2+1)}{z(z-2)}\right\} = \frac{5}{2}$$

$$\oint_c f(z)\, dz = 2\pi i\left[\frac{5}{2} - \frac{1}{2}\right]$$

$$= \underline{\underline{4\pi i}}$$

Q The value of $\oint_c \frac{1-2z}{z(z-1)(z-2)}$ where c is $|z|=1.5$ is ____

Soln here $z=0$ and $z=1.5$ lies inside $|z| = 1.5$ and are simple pole
$z=2$ lies out side

$$\text{Res }(f(z))_{at\ z\to 0} = \lim_{z\to 0}\left\{(z-0)\times\frac{1-2z}{(z-0)(z-1)(z-2)}\right\} = \frac{1}{2}$$

$$\left(\text{Res } f(z)\right)_{z=1} = \lim_{z\to 1} \left\{ (z-1)\frac{1-2z}{z(z+1)(z-2)} \right\} = \frac{-1}{-1} = 1$$

$$\oint_C f(z)\,dz = 2\pi i\left[\frac{1}{2}+1\right]$$

$$= 3\pi i$$

Q The value of $\oint_C \frac{z^2}{(z-1)^2(z+2)}$ where C is $|z|=3$ is ____

here $z=1$ $z=-2$ lies inside $|z|=3$

$z=1$ is a pole of order 2

$z=-2$ is a simple pole

$$\text{Res } f(z)\Big|_{z=-2} = \lim_{z\to -2} \left\{ (z+2)\frac{z^2}{(z-1)^2(z+2)} \right\} = \frac{4}{9}$$

$$\text{Res } f(z)\Big|_{z=1} = \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d}{dz}\left\{ \frac{(z-1)^2 z^2}{(z-1)^2(z+2)} \right\} = \lim_{z\to 1}\left\{ \frac{(z+2)2z - z^2(1)}{(z+2)^2} \right\} = \frac{5}{9}$$

$$\oint_C f(z)\,dz = 2\pi i\left[\frac{4}{9}+\frac{5}{9}\right]$$

$$= 2\pi i$$

Q The value of $\oint_C \tan z\,dz$ where C is $|z|=2$ is ____

Sol^n $\tan z = \frac{\sin z}{\cos z}$ $\quad \cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}$ where $n = 0, \pm 1, \pm 2 \ldots$

when $n=0$ $z = \pi/2 \approx 1.57$

$\frac{0}{0}$ form. $\quad n=-1$ $z = -\pi/2$ are simple poles

$$\text{Res } f(z)\Big|_{z=\pi/2} = \lim_{z\to \pi/2} (z-\pi/2)\frac{\sin z}{\cos z} = \lim_{z\to \pi/2} \frac{(z-\pi/2)(\cos z + \sin z(1))}{-\sin z} = \frac{1}{-1} = -1$$

$$\text{Res } f(z)\Big|_{z=-\pi/2} = \lim_{z\to -\pi/2} (z+\pi/2)\frac{\sin z}{\cos z} = \lim_{z\to -\pi/2} \frac{(z+\pi/2)\cos z + \sin z(1)}{-\sin z} = \frac{-1}{1} = -1$$

$$\oint_C f(z)\,dz = 2\pi i\,[-1-1]$$

$$= -4\pi i$$

Taylor's Series of $f(z)$. The taylor series of $f(z)$ about $z=a$ is given by $f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$

It consist of only +ve powers of $(z-a)$ only.

Laurent Series : The Laurent series of $f(z)$ about $z=a$ is given by

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

It consist of both +ve and -ve powers of $(z-a)$.

Q Find the laurent series of $f(z) = \frac{1}{(z-1)(z+2)}$ about $z=1$

expanding laurent series about singular point.

$$f(z) = \frac{1}{(z-1)(z+2)}$$

$$\boxed{\begin{aligned} (1+x)^{-1} &= 1 - x + x^2 - x^3 \\ (1+x)^{-2} &= 1 - 2x + 3x^2 - \cdots \end{aligned}}$$

$$f(z) = \frac{1}{(z-1)(z+2)} = \frac{1}{(z-1)(z-1+3)} = \frac{1}{(z-1)(3)}\left[1 + \left(\frac{z-1}{3}\right)\right]^{-1}$$

$$= \frac{1}{(z-1)\,3}\left[1 - \frac{(z-1)}{3} + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \cdots\cdots\right]$$

$$= \frac{1}{3}\,\frac{1}{z-1} - \frac{1}{3^2}(1) + \frac{1}{3^3}(z-1) - \frac{1}{3^4}(z-1)^2 + \cdots$$

Residue

same as before page

$$\left[\text{Res } f(z)\right]_{z=1} = \lim_{z\to 1}\left\{(\cancel{z-1})\frac{1}{(\cancel{z-1})(z+2)}\right\} = \frac{1}{3}$$

finding residue by Laurent series takes time so cal^c by using Cauchy formula.

Q Find the Laurent series of $f(z) = \frac{1}{(z-1)^2(z+2)}$ about $z=1$

Soln

$$f(z) = \frac{1}{(z-1)^2(z+2)} = \frac{1}{(z-1)^2(z-1+3)} = \frac{1}{(z-1)^2 3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1}$$

$$= \frac{1}{(z-1)^2(3)}\left[1 - \frac{(z-1)}{3} + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{3}\frac{1}{(z-1)^2} - \frac{1}{3^2}\frac{1}{z-1} + \frac{1}{3^3}(1) - \frac{1}{3^4}(z-1) + \cdots$$

Residue ← same

Now we will again cal^c Residue using Res formula

$$\left[\text{Res } f(z)\right]_{z=1} = \lim_{z\to 1}\frac{1}{\lfloor 2-1}\frac{d}{dz}\left\{(z-1)^2\frac{1}{(z-1)^2(z+2)}\right\} = \lim_{z\to 1}\left[\frac{-1}{(z+2)^2}\right] = \frac{-1}{9}$$

Q Find the Laurent series of $f(z) = \frac{1}{(z^2+1)}$ about $z = i$

Soln

$$f(z) = \frac{1}{(z+i)(z-i)} = \frac{1}{(z-i)(z-i+2i)} = \frac{1}{(z-i)(2i)}\left[1+\left(\frac{z-i}{2i}\right)\right]^{-1}$$

$$= \frac{1}{(z-i)(2i)}\left[1 - \left(\frac{z-i}{2i}\right) + \left(\frac{z-i}{2i}\right)^2 - \left(\frac{z-i}{2i}\right)^3 + \cdots\right]$$

$$= \frac{1}{2i}\frac{1}{(z-i)} - \frac{1}{(2i)^2}(1) + \frac{1}{(2i)^3}(z-i) - \frac{1}{(2i)^4}(z-i)^2 + \cdots$$

Residue ←

$$\left[\text{Res } f(z)\right]_{z=i} = \lim_{z\to i}(z-i)\frac{1}{(z^2+1)} = \frac{1}{2i}$$

~~Use laurent series~~ for singular

Residue: In the Laurent series expansion of $f(z)$ about $z=a$ the coefficient of $(z-a)^{-1}$ is called the residue.

In the laurent's series expansion of $f(z)$ about $z=a$

(1) If there exists only one term with -ve power of $(z-a)$ [i.e $(z-a)^{-1}$ term only] then $z=a$ is called a simple pole.

(2) If there exist n terms with -ve power of $(z-a)$ [i.e upto $(z-a)^{-n}$ terms] then $z=a$ is called a pole of order 'n'.

(3) If there exist ∞ terms with -ve powers of $z-a$ then $z=a$ is called an essential singularity

(4) If there exist no terms with -ve power of $(z-a)$ then $z=a$ is called a removable singularity.

Q The value of $\oint_{|z|=3} \sin\left(\frac{1}{z-2}\right) dz$

here $z=2$ is an essential singularity so we will have ~~have~~ use Laurent se..

$$\sin\left(\frac{1}{z-2}\right) = \underbrace{1}_{\text{Residue}}\frac{1}{z-2} - \frac{1}{3!}\frac{1}{(z-2)^3} + \frac{1}{5!(z-2)^5} - \frac{1}{\;} \dots$$

$$\oint_C f(z)\,dz = 2\pi i[1]$$

Q. The value of $\oint_{|z|=3} (z-2)^2 \sin\left(\frac{1}{z-2}\right) dz$

here $z=2$ is essential singularity

$$= (z-2)^2 \left[\frac{1}{z-2} - \frac{1}{3!}\frac{1}{(z-2)^3} + \frac{1}{5!(z-2)^5} \cdots\cdots\right]$$

$$= (z-2) \left(-\frac{1}{3!}\right)\frac{1}{(z-2)} + \frac{1}{5!(z-2)^3}$$

Residue $= \frac{-1}{6}$

$$\oint_C f(z)\,dz = 2\pi i\left[-\frac{1}{3!}\right] = -\frac{\pi i}{3}$$

Q. The value of $\oint_{|z|=3} \cos\left(\frac{1}{z-2}\right) dz$

here $z=2$ is an essential singularity.

$$\cos\left(\frac{1}{z-2}\right) = 1 - \frac{1}{2!}\frac{1}{(z-2)^2} + \frac{1}{4!(z-2)^4} - \cdots\cdots + 0.\frac{1}{(z-2)}$$

Residue

$$\oint_C f(z)\,dz = 2\pi i[0] = 0$$

converse of Cauchy Integral theorem need not be true.

$\oint_C f(z) = 0$ — it does not mean that $f(z)$ is analytic func

यहाँ func $\cos\left(\frac{1}{z-2}\right)$ जो analytic नहीं है $z=2$ पे ~~पर~~ फिर भी and इसका residue 0 आ रहा है $z=2$ पे तो $\oint f(z) = \text{or } 2\pi i[0] = 0$ तो मतलब कोई ऐसा सोच सकता है कि $\oint_C f(z) = 0$ है तो $f(z)$ analytic होना ही चाहिए तो वे सोचना गलत होगा।

Q. The value of $\oint_{|z|=1} z^3 e^{1/z}\, dz$

Here $z=0$ is an essential singularity

$$z^3 e^{1/z} = z^3\left[1+\frac{1}{z}+\frac{1}{z^2 2!}+\frac{1}{z^3 3!}+\frac{1}{z^4 4!}+\cdots\right]$$

$$= z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{z\,4!} + \cdots$$

Residue

$$\oint_c f(z)\,dz = 2\pi i\left[\frac{1}{4!}\right] = \frac{\pi i}{12}$$

Q. The Residue of $f(z)=\dfrac{\sin z}{z^6}$ about its singularity is ____

$$\left(\text{Res } f(z)\right)_{z=0} = \lim_{z\to 0} \frac{1}{5!}\frac{d^5}{dz^5}\left[z^6 \cdot \frac{\sin z}{z^6}\right]$$

$$= \lim_{z\to 0} \frac{1}{5!}\cos z$$

$$= \frac{1}{5!} = \frac{1}{120}$$

$$\frac{\sin z}{z^6} = \frac{1}{z^6}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^5} - \frac{1}{z^3 3!} + \frac{1}{5!\,z} - \frac{z}{7!}$$

Residue

Laurent series not only expanded about its singularity but about some region also.

Q The Laurents series of $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$ in the region ① $|z|<1$

② $1<|z|<2$

③ $|z|>2$

Sol^n In the region $|z|<1 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{-1[1-z]} + \frac{1}{2[1-z/2]}$$

$$= -1\,[1-z]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= -1\left[1+z+z^2+\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

② In the region $1<|z|<2 \Rightarrow \frac{1}{|z|}<1$ & $\frac{|z|}{2}<1$

$$f(z) = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-\frac{z}{2})} = \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

③ In the region $|z|>2 \Rightarrow \frac{2}{|z|}<1$, $\frac{1}{|z|}<1$

$$f(z) = \frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-2/z)} = \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right] - \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

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